On the Delay and Gaussian Random Scattering Estimation for GNSS-R Applications

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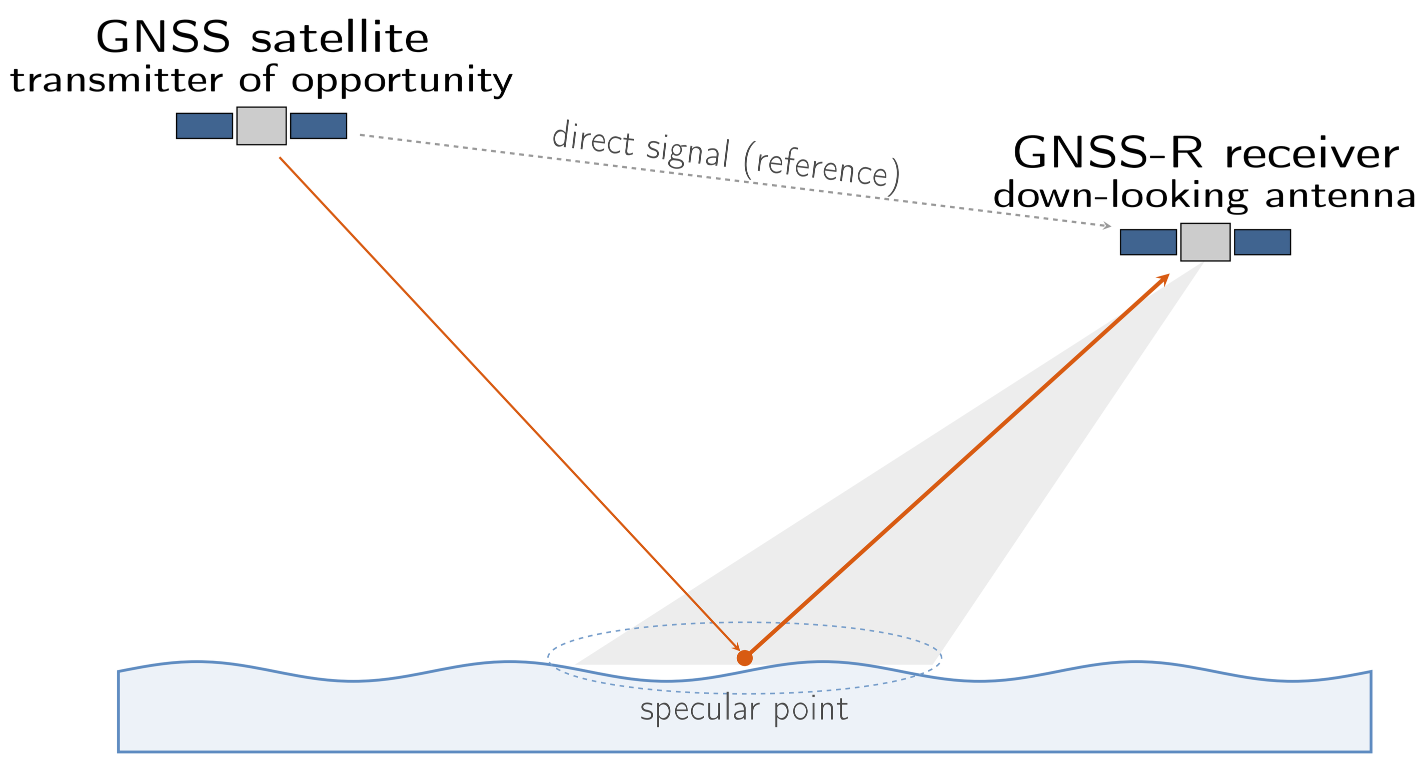
Motivation & Contributions

- GNSS-R retrievals (altimetry, wind, soil moisture, sea ice) are driven by the **delay–Doppler map (DDM)**; standard processing assumes a **conditional signal model (CSM)** where everything is deterministic ⇒ *no direct access to the diffuse scattering power*.

Contributions

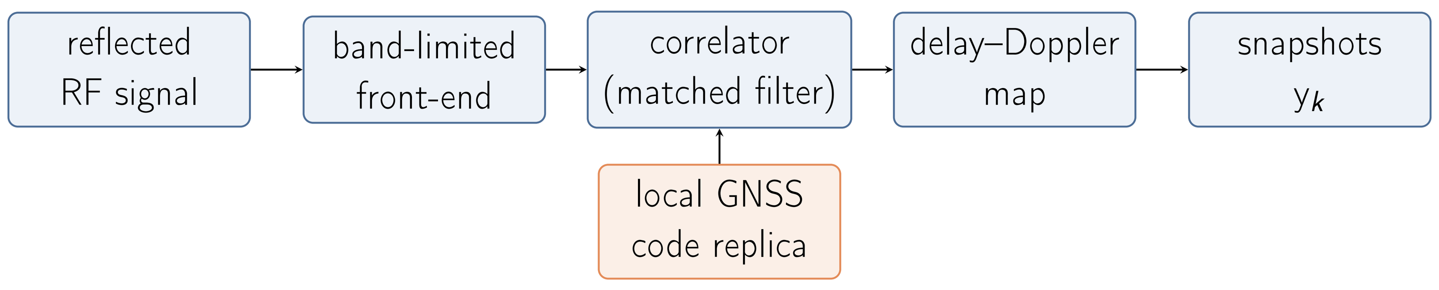
- Closed-form **unconditional CRB** (CRB_u) for delay τ, scattering mean $\mu_\alpha = \rho e^{j\varphi}$, variance σ_α^2, and noise power σ_n^2.
- Corresponding **unconditional MLE** (UMLE) expressions.
- Validation with a representative band-limited GNSS signal (GPS L1 C/A).

GNSS-R: Sensing the Surface with Signals of Opportunity



- A **smooth** surface (calm sea, ice, wet soil) reflects **coherently**: strong specular component μ_α.
- A **rough** surface scatters **diffusely** over the glistening zone: random component of power σ_α^2.
- The coherent/diffuse balance is itself a **surface roughness observable** ⇒ worth estimating, not discarding.

From Reflected Signal to Statistical Model



K snapshots of N samples at $F_s \geq B$ (Doppler known and compensated):

$$\mathrm{y}_k = \alpha_k \, \mathrm{s}(\tau) + \mathrm{n}_k, \qquad \mathrm{n}_k \sim \mathcal{CN}(0, \sigma_n^2 \mathrm{I}_N), \qquad k = 1, \ldots, K,$$

where the reflection coefficient carries the surface physics:

$$\alpha_k = \underbrace{\mu_\alpha}_{\text{coherent / specular}} + \underbrace{u_k}_{\text{diffuse fluctuation}}, \qquad u_k \sim \mathcal{CN}(0, \sigma_\alpha^2), \quad \mu_\alpha = \rho e^{j\varphi}.$$

CSM (standard): α_k deterministic per snapshot — unknowns σ_n^2, $\boldsymbol{\alpha}$, τ.

USM (this work): Gaussian prior $\alpha_k \sim \mathcal{CN}(\mu_\alpha, \sigma_\alpha^2)$, so

$$\mathrm{y}_k \sim \mathcal{CN}\big(\mu_\alpha \mathrm{s}(\tau),\ \sigma_\alpha^2 \mathrm{s}(\tau)\mathrm{s}(\tau)^H + \sigma_n^2 \mathrm{I}_N\big), \qquad \boldsymbol{\theta}^T = (\sigma_n^2,\ \sigma_\alpha^2,\ \rho,\ \varphi,\ \tau).$$

Unconditional MLE (UMLE)

With $r_k(\tau) = \mathrm{s}(\tau)^H \mathrm{y}_k$ and $\bar{r}(\tau) = \frac{1}{K}\sum_k r_k(\tau)$. The standard **CMLE** is $\widehat{\tau}_c = \arg\max_\tau \sum_k |r_k(\tau)|^2$; scattering mean/variance are only *indirectly* available as post-hoc sample statistics of $\widehat{\alpha}_{k,c} = r_k(\widehat{\tau}_c)/\|\mathrm{s}\|^2$.

Minimizing the negative log-likelihood of the USM instead yields:

$$\widehat{\tau}_u = \arg\min_\tau\ C_u(\tau),\quad C_u(\tau) = \left(\sum_{k=1}^{K}\left(\|\mathrm{y}_k\|^2 - \frac{|r_k(\tau)|^2}{\|\mathrm{s}\|^2}\right)\right)^{N-1}\left(\sum_{k=1}^{K}\left|r_k(\tau) - \bar{r}(\tau)\right|^2\right),$$

and then

$$\widehat{\sigma}^2_{n,u} = \frac{1}{K(N-1)}\sum_{k=1}^{K}\left(\|\mathrm{y}_k\|^2 - \frac{|r_k(\widehat{\tau}_u)|^2}{\|\mathrm{s}\|^2}\right), \qquad \widehat{\mu}_{\alpha,u} = \frac{\bar{r}(\widehat{\tau}_u)}{\|\mathrm{s}\|^2},$$

$$\widehat{\sigma}^2_{\alpha,u} = \frac{1}{K(N-1)}\sum_{k=1}^{K}\left(N\frac{|r_k(\widehat{\tau}_u)|^2}{\|\mathrm{s}\|^4} - \frac{\|\mathrm{y}_k\|^2}{\|\mathrm{s}\|^2}\right) - \frac{|\widehat{\mu}_{\alpha,u}|^2}{\|\mathrm{s}\|^2}.$$

- σ_α^2 is estimated *directly* within the model — not as a post-hoc sample variance.

Closed-Form Unconditional CRB

Slepian–Bangs formula on the USM ⇒ the FIM is **block-diagonal**: the powers $(\sigma_n^2, \sigma_\alpha^2)$ decouple from ρ and from the pair (φ, τ). With $\beta = \sigma_\alpha^2\|\mathrm{s}\|^2/\sigma_n^2$ (diffuse SNR), $\gamma = \rho^2\|\mathrm{s}\|^2/\sigma_n^2$ (coherent SNR), and the band-limited signal terms $w_1 = \|\mathrm{s}\|^2/F_s$, $w_3 = \mathrm{s}^H\boldsymbol{\Lambda}\mathrm{s}/F_s$, $W_{33} = F_s\,\mathrm{s}^H\mathrm{V}\mathrm{s}$:

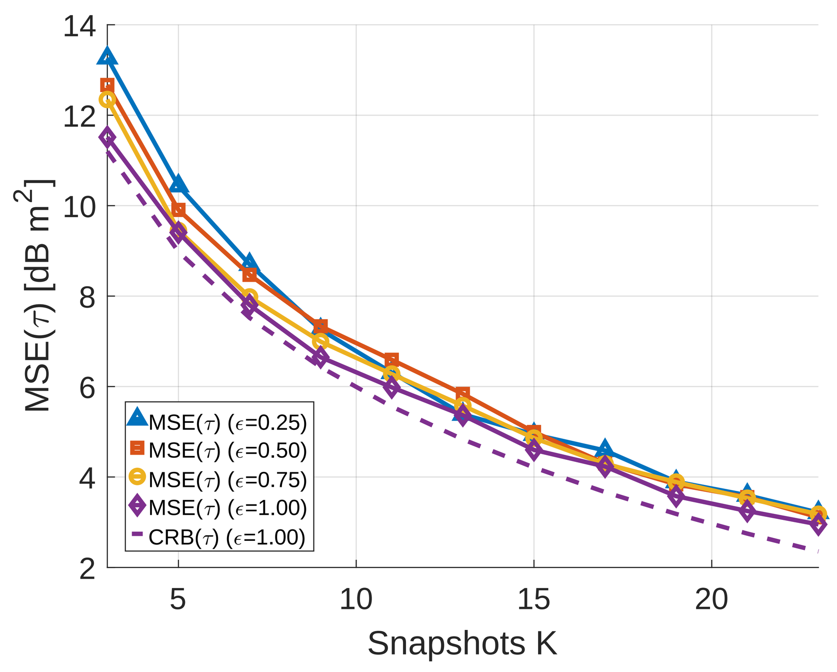
Resulting closed-form bounds:

$$\text{CRB}_u(\tau) = \frac{1+\beta}{2K\left(\gamma + \beta(\gamma+\beta)\right)\left(\frac{W_{33}}{w_1} - \frac{|w_3|^2}{w_1^2}\right)},$$

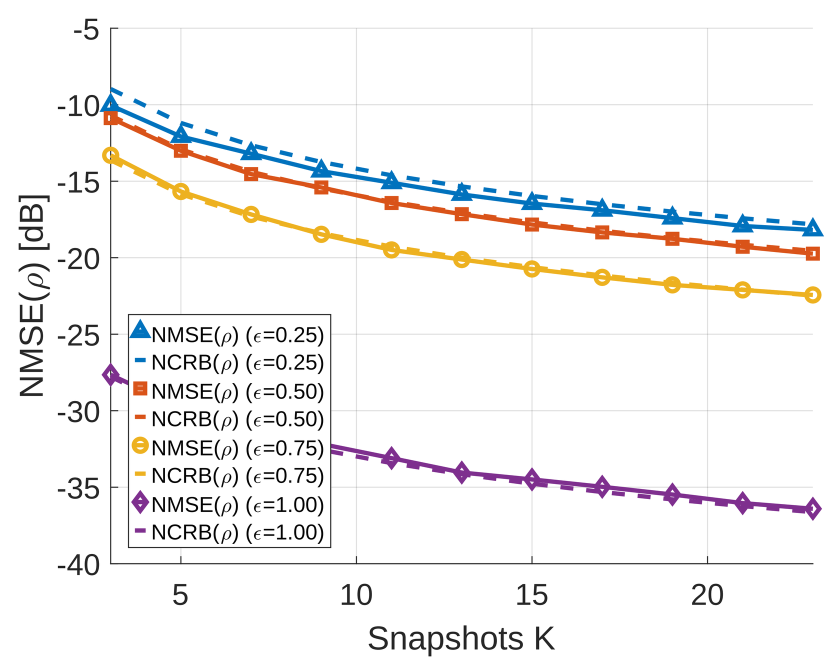
$$\text{CRB}_u(\sigma_\alpha^2) = \frac{(\sigma_\alpha^2)^2\left(1 + (N-1)(1+\beta)^2\right)}{K(N-1)\beta^2}, \qquad \text{CRB}_u(\rho) = \frac{1+\beta}{2K\gamma}\sigma_\alpha^2.$$

- Compact expressions in terms of physically interpretable quantities (β, γ, signal bandwidth terms).
- Output SNR: $\text{SNR}_{\text{out}} = P\|\mathrm{s}\|^2/\sigma_n^2$, total power $P = \rho^2 + \sigma_\alpha^2$, **coherent power fraction** $\epsilon = \rho^2/P \in (0,1)$: $\epsilon = 0$: zero-mean USM (fully diffuse) $\longleftrightarrow$ $\epsilon = 1$: coherent CSM.

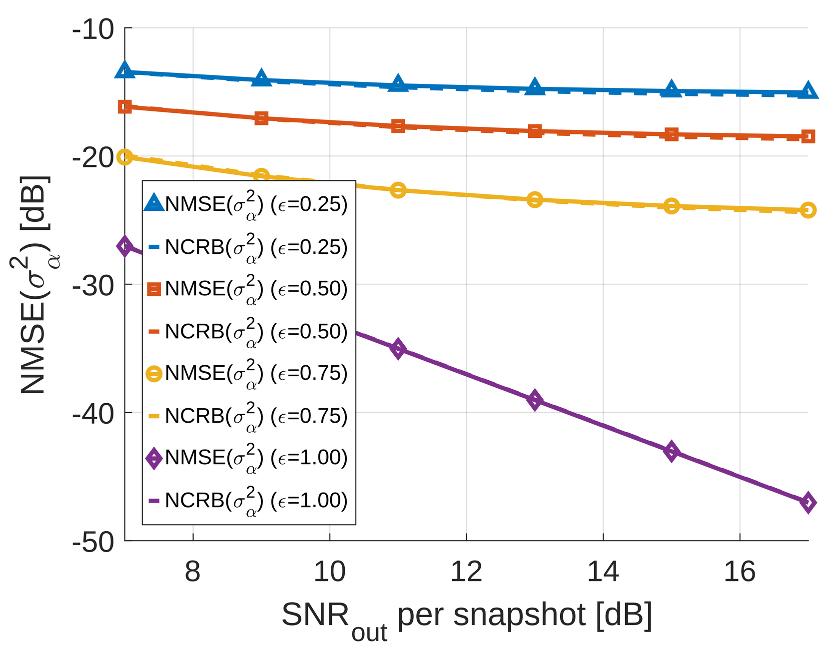
Validation: UMLE vs. CRB_u



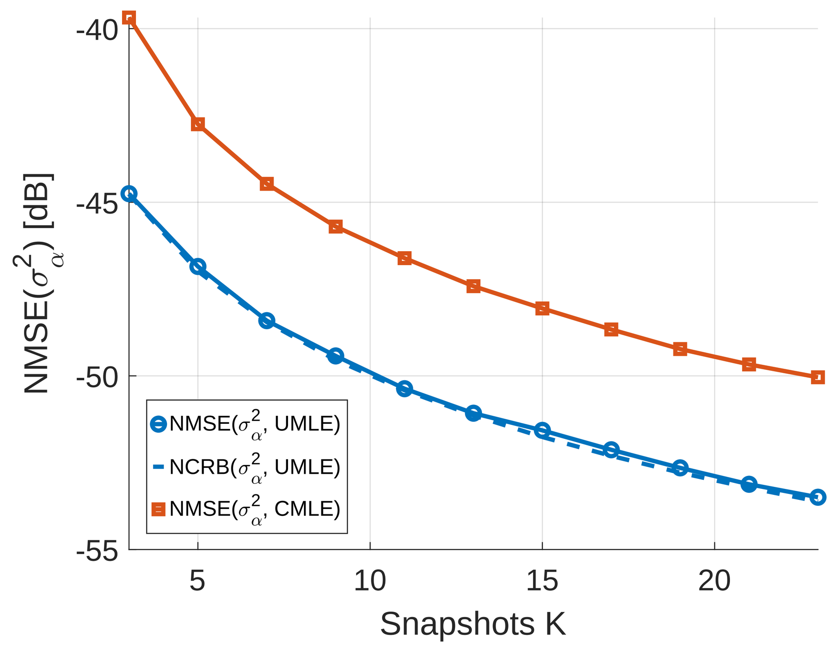
MSE of $\widehat{\tau}_u$ vs. K ($\text{SNR}_{\text{out}} = 20$ dB), for several ϵ.



NMSE of $\widehat{\rho}$ vs. K: larger ϵ (more coherent) ⇒ better estimation.



NMSE of $\widehat{\sigma}_n^2$ vs. SNR_{out} ($K = 20$).



σ_α^2 estimation when $\sigma_\alpha^2 = 0$ ($\epsilon = 1$, $\text{SNR}_{\text{out}} = 20$ dB).

- Simulated **GPS L1 C/A** signal sampled at $F_s = 4$ MHz; 1000 Monte Carlo runs.
- All MSEs attain the proposed CRB_u ⇒ **b "UMLE asymptotically efficient for large K (and SNR), CRB_u validated.**
- Bottom right: the **UMLE outperforms the CMLE sample variance** for the estimation of a small σ_α^2.

Takeaways

- The **UMLE preserves the delay estimation performance** of the CMLE
- while giving **direct access to more information**: σ_α^2 (not observable in the CSM) and ρ, φ (not observable in the zero-mean USM).
- Closed-form CRB_u = ultimate achievable performance for delay + Gaussian scattering mean/variance estimation; UMLEs shown asymptotically efficient.
- Outlook: relevant for coherence detection/characterization and decoupling diffuse/coherent contributions in DDM-based retrievals (sea-ice detection, soil moisture, altimetry).

References & Acknowledgments

[1] V. Zavorotny et al., "Tutorial on remote sensing using GNSS bistatic radar of opportunity," *IEEE GRSM*, 2014.
[2] P. Stoica, A. Nehorai, "Performance study of conditional and unconditional direction-of-arrival estimation," *IEEE TASSP*, 1990.
[3] P. Das, J. Vilà-Valls et al., "On the accuracy limit of time-delay estimation with a band-limited signal," *ICASSP*, 2020.
[4] A. Renaux et al., "On the high-SNR conditional maximum-likelihood estimator full statistical characterization," *IEEE TSP*, 2007.
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