

Technical Note

Details on SIGMA integrals evaluation

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The aim of this note is to provide details on the evaluation of the following integrals:

$$\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v, \quad (1)$$

$$\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} \left(u \ln \left(\frac{u}{v} \right) \right)^2 p(u)p(v) \mathrm{d}u \mathrm{d}v, \quad (2)$$

$$\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u^2 \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v, \quad (3)$$

$$\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} uv \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v, \quad (4)$$

$$\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} \int_{w=0}^{+\infty} uw \left| \ln \left(\frac{u}{v} \right) \right| \left| \ln \left(\frac{w}{u} \right) \right| p(u)p(v)p(w) \mathrm{d}u \mathrm{d}v \mathrm{d}w, \quad (5)$$

where $p(\cdot)$ refers to the probability density function of a χ^2 distributed random variable.

$$p(x) = \frac{1}{2} \exp \left(-\frac{x}{2} \right). \quad (6)$$

1 Preliminary results

In order to evaluate (1)–(5), a number of intermediate integrals need to be considered.

1.1 Euler-Mascheroni constant

A first integral is the the following:

$$\int_{\mathbb{R}^+} \ln(x) e^{-x} \mathrm{d}x = \lim_{\mu \rightarrow 0} \frac{\mathrm{d}}{\mathrm{d}\mu} \int_{\mathbb{R}^+} x^\mu e^{-x} \mathrm{d}x \quad (7)$$

$$= \lim_{\mu \rightarrow 0} \frac{\mathrm{d}}{\mathrm{d}\mu} \Gamma(\mu + 1) \quad (8)$$

$$= \lim_{\mu \rightarrow 0} \Gamma(\mu + 1) \psi(\mu + 1) \quad (9)$$

$$= \Gamma(1) \psi^{(0)}(1) = -\gamma, \quad (10)$$

where $\psi^{(n)}(x)$ est the polygamma function, defined as the $n + 1$ -th derivative of $\ln(\Gamma(x))$ [1, Sec. 6.3] and $\gamma = 0.577215\dots$ is the Euler-Mascheroni constant. Similarly,

$$\int_{\mathbb{R}^+} \ln(x)^2 e^{-x} dx = \lim_{\mu \rightarrow 0} \frac{d^2}{d\mu^2} \int_{\mathbb{R}^+} x^\mu e^{-x} dx \quad (11)$$

$$= \lim_{\mu \rightarrow 0} \frac{d^2}{d\mu^2} \Gamma(\mu + 1) \quad (12)$$

$$= \lim_{\mu \rightarrow 0} \Gamma(\mu + 1) ((\psi^{(0)}(\mu + 1))^2 + \psi^{(1)}(\mu + 1)) \quad (13)$$

$$= \Gamma(1) ((\psi^{(0)}(1))^2 + \psi^{(1)}(1)) = \gamma^2 + \frac{\pi^2}{6}. \quad (14)$$

1.2 Exponential integral Ei

In the following computation the exponential integral $\text{Ei}(\cdot)$ is often used. It is defined as

$$\text{Ei}(x) = - \int_{-x}^{+\infty} \frac{e^{-x}}{x} dx. \quad (15)$$

An important property of this integral is its expansion around 0, for $x > 0$:

$$\text{Ei}\left(-\frac{x}{2}\right) = \ln(x) - \ln(2) + \gamma + O(x). \quad (16)$$

Besides, it appears in integrals involving $\ln$ and $\exp$ functions.

$$\int_a^b \ln(x) e^{-x/2} dx = [-2e^{-x/2} \ln(x)]_a^b - \int_a^b -2 \frac{e^{-x/2}}{x} dx \quad (17)$$

$$= [-2e^{-x/2} \ln(x)]_a^b - \int_{a/2}^{b/2} -2 \frac{e^{-y}}{y} dy \quad (18)$$

$$= [-2e^{-x/2} \ln(x)]_a^b + 2 \left(\int_{a/2}^{+\infty} \frac{e^{-y}}{y} dy - \int_{b/2}^{+\infty} \frac{e^{-y}}{y} dy \right) \quad (19)$$

$$= [-2e^{-x/2} \ln(x)]_a^b + 2 \left(-\text{Ei}\left(-\frac{a}{2}\right) + \text{Ei}\left(-\frac{b}{2}\right) \right) \quad (20)$$

$$= \left[2\text{Ei}\left(-\frac{x}{2}\right) - 2e^{-x/2} \ln(x) \right]_a^b. \quad (21)$$

Thanks to (16), a special case for $a = 0$ yields:

$$\lim_{a \rightarrow 0^+} 2\text{Ei}\left(-\frac{x}{2}\right) - 2e^{-x/2} \ln(x) = \lim_{a \rightarrow 0^+} 2\ln(x) - 2\ln(2) + 2\gamma - 2\ln(x) = -2(\ln(2) - \gamma) \quad (22)$$

The following expressions are obtained from [2, Sec. 1.3.2, eq (1) and (3)].

$$\int x e^{-x/2} \text{Ei}\left(-\frac{x}{2}\right) dx = 4\text{Ei}(-x) - 2(x+2)e^{-x/2} \text{Ei}\left(-\frac{x}{2}\right) - 2e^{-x}, \quad (23)$$

$$\int x^2 e^{-x/2} \text{Ei}\left(-\frac{x}{2}\right) dx = 16\text{Ei}(-x) - 2(x^2 + 4x + 8)e^{-x/2} \text{Ei}\left(-\frac{x}{2}\right) - 2(x+5)e^{-x}, \quad (24)$$

$$\int x e^{-x} \text{Ei} \left(-\frac{x}{2} \right) dx = \text{Ei} \left(-\frac{3x}{2} \right) - (x+1) e^{-x} \text{Ei} \left(-\frac{x}{2} \right) - \frac{2}{3} e^{-3x/2}, \quad (25)$$

from [2, Sec. 2.5.10 eq (6)]

$$\int_0^{+\infty} x \ln(x) e^{-x/2} \text{Ei} \left(-\frac{x}{2} \right) dx = -2 \ln(2)^2 + 4(\gamma - 1) \ln(2) - 2\gamma + \frac{\pi^2}{3}, \quad (26)$$

from [2, Sec. 2.5.12 eq (8)]

$$\int_0^{+\infty} x e^{-x/2} \text{Ei} \left(-\frac{x}{2} \right)^2 dx = -4 \ln(3)^2 - 4 \ln(3) + 8 \ln(2) \ln(3) - 8 \text{Li}_2 \left(\frac{1}{3} \right) + \frac{2\pi^2}{3}, \quad (27)$$

where $\text{Li}_2(x)$ denotes the polylogarithm, defined as

$$\text{Li}_n(x) = \sum_{k=1}^{+\infty} \frac{x^k}{k^n}. \quad (28)$$

A numerical evaluation of the specific value of interest yields: $\text{Li}_2 \left(\frac{1}{3} \right) \approx 0.366213$.

Finally, from [2, Sec. 1.3.1 eq (11)]

$$\begin{aligned} \int_0^{+\infty} \frac{1}{x} \text{Ei} \left(-\frac{x}{2} \right)^2 dx &= \lim_{\epsilon \rightarrow 0} -\frac{1}{2} \left((\ln(\epsilon) - \ln(2) + \gamma)^2 + \zeta(2) \right) - \sum_{k=1}^{+\infty} \frac{(-1)^k \epsilon^k}{k! k^2 2^k} \\ &= \lim_{\epsilon \rightarrow 0} -\frac{1}{2} \left((\ln(\epsilon))^2 - 2(\ln(2) - \gamma) \ln(\epsilon) + (\ln(2) - \gamma)^2 + \frac{\pi^2}{6} \right). \end{aligned} \quad (29)$$

1.3 Intermediate integrals

Using a simple change of variable,

$$\int_{\mathbb{R}^+} \ln(u) e^{-u/2} du = 2 \ln(2) \int_{\mathbb{R}^+} e^{-x} dx + 2 \underbrace{\int_{\mathbb{R}^+} \ln(x) e^{-x} dx}_{(10)} \quad (30)$$

$$= 2 \ln(2) - 2\gamma. \quad (31)$$

$$\boxed{\int_{\mathbb{R}^+} \ln(u) e^{-u/2} du = 2(\ln(2) - \gamma)} \quad (32)$$

Using the integration by parts method,

$$\int_{\mathbb{R}^+} u \ln(u) e^{-u/2} du = [u \ln(u) (-2e^{-u/2})]_0^{+\infty} - \int_{\mathbb{R}^+} (\ln(u) + 1) (-2e^{-u/2}) du \quad (33)$$

$$= 2 \underbrace{\int_{\mathbb{R}^+} \ln(u) e^{-u/2} du}_{(32)} + 2 \underbrace{\int_{\mathbb{R}^+} e^{-u/2} du}_{=2} \quad (34)$$

$$= 4(\ln(2) - \gamma) + 4. \quad (35)$$

$$\boxed{\int_{\mathbb{R}^+} u \ln(u) e^{-u/2} du = 4 + 4(\ln(2) - \gamma)} \quad (36)$$

Again, using a change of variable,

$$\int_{\mathbb{R}^+} \ln(u)^2 e^{-u/2} du = 2 \ln(2)^2 \int_{\mathbb{R}^+} e^{-x} dx + 4 \ln(2) \underbrace{\int_{\mathbb{R}^+} \ln(x) e^{-x} dx}_{(10)} + 2 \underbrace{\int_{\mathbb{R}^+} \ln(x)^2 e^{-x} dx}_{(14)} \quad (37)$$

$$= 2(\ln(2))^2 - 4 \ln(2) \gamma + 2 \left(\gamma^2 + \frac{\pi^2}{6} \right). \quad (38)$$

$$\boxed{\int_{\mathbb{R}^+} \ln(u)^2 e^{-u/2} du = 2(\ln(2) - \gamma)^2 + \frac{\pi^2}{3}} \quad (39)$$

By using the integration by parts method,

$$\int_{\mathbb{R}^+} u^2 \ln(u) e^{-u/2} du = [u^2 \ln(u) (-2e^{-u/2})]_0^{+\infty} - \int_{\mathbb{R}^+} (2u \ln(u) + u) (-2e^{-u/2}) du \quad (40)$$

$$= 4 \underbrace{\int_{\mathbb{R}^+} u \ln(u) e^{-u/2} du}_{(36)} + 2 \underbrace{\int_{\mathbb{R}^+} u e^{-u/2} du}_{=4} \quad (41)$$

$$= 16 + 16(\ln(2) - \gamma) + 8. \quad (42)$$

$$\boxed{\int_{\mathbb{R}^+} u^2 \ln(u) e^{-u/2} du = 24 + 16(\ln(2) - \gamma)} \quad (43)$$

$$\int_{\mathbb{R}^+} u \ln(u)^2 e^{-u/2} du = [u \ln(u)^2 (-2e^{-u/2})]_0^{+\infty} - \int_{\mathbb{R}^+} (\ln(u)^2 + 2 \ln(u)) (-2e^{-u/2}) du \quad (44)$$

$$= 2 \underbrace{\int_{\mathbb{R}^+} \ln(u)^2 e^{-u/2} du}_{(39)} + 4 \underbrace{\int_{\mathbb{R}^+} \ln(u) e^{-u/2} du}_{(32)} \quad (45)$$

$$= 4(\ln(2) - \gamma)^2 + \frac{2\pi^2}{3} + 8(\ln(2) - \gamma). \quad (46)$$

$$\boxed{\int_{\mathbb{R}^+} u \ln(u)^2 e^{-u/2} du = 4(\ln(2) - \gamma)^2 + 8(\ln(2) - \gamma) + \frac{2\pi^2}{3}} \quad (47)$$

$$\int_{\mathbb{R}^+} u^2 \ln(u)^2 e^{-u/2} du = [u^2 \ln(u)^2 (-2e^{-u/2})]_0^{+\infty} - \int_{\mathbb{R}^+} (2u \ln(u)^2 + 2u \ln(u)) (-2e^{-u/2}) du \quad (48)$$

$$= 4 \underbrace{\int_{\mathbb{R}^+} u \ln(u)^2 e^{-u/2} du}_{(47)} + 4 \underbrace{\int_{\mathbb{R}^+} u \ln(u) e^{-u/2} du}_{(36)} \quad (49)$$

$$= 16(\ln(2) - \gamma)^2 + 32(\ln(2) - \gamma) + \frac{8\pi^2}{3} + 16 + 16(\ln(2) - \gamma). \quad (50)$$

$$\boxed{\int_{\mathbb{R}^+} u^2 \ln(u)^2 e^{-u/2} du = 16(\ln(2) - \gamma)^2 + 48(\ln(2) - \gamma) + 16 + \frac{8\pi^2}{3}} \quad (51)$$

2 Evaluation of integral (1)

$$\begin{aligned}
& \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v \\
&= \int_{u=0}^{+\infty} u p(u) \left(\underbrace{\int_{v=0}^u \ln \left(\frac{u}{v} \right) p(v) \mathrm{d}v}_{\text{(a)}} - \underbrace{\int_{v=u}^{+\infty} \ln \left(\frac{u}{v} \right) p(v) \mathrm{d}v}_{\text{(b)}} \right) \mathrm{d}u
\end{aligned} \tag{52}$$

For (a) of (52): using (6),

$$\int_{v=0}^u \ln \left(\frac{u}{v} \right) p(v) \mathrm{d}v = \frac{1}{2} \left(\ln(u) \int_{v=0}^u e^{-v/2} \mathrm{d}v - \underbrace{\int_{v=0}^u \ln(v) e^{-v/2} \mathrm{d}v}_{\text{(21)}} \right) \tag{53}$$

$$= \frac{1}{2} \left(2 \ln(u) (1 - e^{-u/2}) - \underbrace{\left[2 \text{Ei} \left(-\frac{v}{2} \right) - 2 \ln(v) e^{-v/2} \right]_0^u}_{\text{(22)}} \right) \tag{54}$$

$$= \ln(u) - \ln(u) e^{-u/2} - \left(\text{Ei} \left(-\frac{u}{2} \right) - \ln(u) e^{-u/2} + (\ln(2) - \gamma) \right) \tag{55}$$

$$= \ln(u) - \text{Ei} \left(-\frac{u}{2} \right) - (\ln(2) - \gamma). \tag{56}$$

where $\text{Ei}(\cdot)$ is the Exponential integral [1, Sec. 5]

For (b) of (52):

$$\int_{v=u}^{+\infty} \ln \left(\frac{u}{v} \right) p(v) \mathrm{d}v = \frac{1}{2} \left(\ln(u) \int_{v=u}^{+\infty} e^{-v/2} \mathrm{d}v - \underbrace{\int_{v=u}^{+\infty} \ln(v) e^{-v/2} \mathrm{d}v}_{\text{(21)}} \right) \tag{57}$$

$$= \frac{1}{2} \left(2 \ln(u) e^{-u/2} - \left[2 \text{Ei} \left(-\frac{v}{2} \right) - 2 \ln(v) e^{-v/2} \right]_u^{+\infty} \right) \tag{58}$$

$$= \ln(u) e^{-u/2} - \left(-\text{Ei} \left(-\frac{u}{2} \right) + \ln(u) e^{-u/2} \right) \tag{59}$$

$$= \text{Ei} \left(-\frac{u}{2} \right) \tag{60}$$

Gathering (a) and (b), one gets

$$\begin{aligned}
& \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v \\
&= \frac{1}{2} \int_{\mathbb{R}^+} u e^{-u/2} \left(\ln(u) - 2\mathrm{Ei} \left(-\frac{u}{2} \right) - (\ln(2) - \gamma) \right) \mathrm{d}u \tag{61}
\end{aligned}$$

$$= \frac{1}{2} \underbrace{\int_{\mathbb{R}^+} u \ln(u) e^{-u/2} \mathrm{d}u}_{(36)} - \underbrace{\int_{\mathbb{R}^+} \mathrm{Ei} \left(-\frac{u}{2} \right) u e^{-u/2} \mathrm{d}u}_{(23)} - \frac{1}{2} (\ln(2) - \gamma) \underbrace{\int_{\mathbb{R}^+} u e^{-u/2} \mathrm{d}u}_{=4} \tag{62}$$

$$= 2 + 2(\ln(2) - \gamma) - \underbrace{\left[4\mathrm{Ei}(-u) - 2(u+2)e^{-u/2}\mathrm{Ei} \left(-\frac{u}{2} \right) - 2e^{-u} \right]_0^{+\infty}}_{(16)} - 2(\ln(2) - \gamma) \tag{63}$$

$$= 2 - 2 + \lim_{\epsilon \rightarrow 0} (4\gamma + 4\ln(\epsilon) - 4(\ln(\epsilon) - \ln(2) + \gamma)) = 4\ln(2). \tag{64}$$

Hence the result,

$$\boxed{\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v = 4\ln(2) \approx 2.772589} \tag{65}$$

3 Evaluation of integral (2)

$$\begin{aligned} & \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} \left(u \ln \left(\frac{u}{v} \right) \right)^2 p(u)p(v) \mathbf{d}u \mathbf{d}v \\ &= \frac{1}{4} \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u^2 (\ln(u)^2 - 2 \ln(u) \ln(v) + \ln(v)^2) e^{-u/2} e^{-v/2} \mathbf{d}u \mathbf{d}v \end{aligned} \quad (66)$$

$$\begin{aligned} &= \frac{1}{4} \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u^2 \ln(u)^2 e^{-u/2} e^{-v/2} \mathbf{d}u \mathbf{d}v - \frac{1}{2} \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u^2 \ln(u) \ln(v) e^{-u/2} e^{-v/2} \mathbf{d}u \mathbf{d}v \\ &\quad + \frac{1}{4} \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u^2 \ln(v)^2 e^{-u/2} e^{-v/2} \mathbf{d}u \mathbf{d}v \end{aligned} \quad (67)$$

$$\begin{aligned} &= \frac{1}{4} \underbrace{\int_{u=0}^{+\infty} u^2 \ln(u)^2 e^{-u/2} \mathbf{d}u}_{(51)} \underbrace{\int_{v=0}^{+\infty} e^{-v/2} \mathbf{d}v}_{=2} - \frac{1}{2} \underbrace{\int_{u=0}^{+\infty} u^2 \ln(u) e^{-u/2} \mathbf{d}u}_{(43)} \underbrace{\int_{v=0}^{+\infty} \ln(v) e^{-v/2} \mathbf{d}v}_{(32)} \\ &\quad + \frac{1}{4} \underbrace{\int_{u=0}^{+\infty} u^2 e^{-u/2} \mathbf{d}u}_{=16} \underbrace{\int_{v=0}^{+\infty} \ln(v)^2 e^{-v/2} \mathbf{d}v}_{(39)} \end{aligned} \quad (68)$$

$$\begin{aligned} &= \frac{1}{2} \left(16(\ln(2) - \gamma)^2 + 48(\ln(2) - \gamma) + 16 + \frac{8\pi^2}{3} \right) - (24 + 16(\ln(2) - \gamma))(\ln(2) - \gamma) \\ &\quad + 8(\ln(2) - \gamma)^2 + \frac{4\pi^2}{3} \end{aligned} \quad (69)$$

$$\begin{aligned} &= 8(\ln(2) - \gamma)^2 + 24(\ln(2) - \gamma) + 8 + \frac{4\pi^2}{3} - 24(\ln(2) - \gamma) - 16(\ln(2) - \gamma)^2 \\ &\quad + 8(\ln(2) - \gamma)^2 + \frac{4\pi^2}{3} \end{aligned} \quad (70)$$

$$= 8 + \frac{8\pi^2}{3} \quad (71)$$

Hence the result,

$$\boxed{\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} \left(u \ln \left(\frac{u}{v} \right) \right)^2 p(u)p(v) \mathbf{d}u \mathbf{d}v = 8 + \frac{8\pi^2}{3} \approx 34.318945} \quad (72)$$

4 Evaluation of integral (3)

$$\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u^2 \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v \quad (73)$$

$$= \int_{u=0}^{+\infty} u^2 p(u) \underbrace{\left(\int_{v=0}^{+\infty} \ln \left(\frac{u}{v} \right) p(v) \mathrm{d}v \right)}_{\text{same as in (52)}} \mathrm{d}u \quad (74)$$

$$= \frac{1}{2} \int_{\mathbb{R}^+} u^2 e^{-u/2} \left(\ln(u) - 2\mathrm{Ei} \left(-\frac{u}{2} \right) - (\ln(2) - \gamma) \right) \mathrm{d}u \quad (75)$$

$$= \frac{1}{2} \underbrace{\int_{\mathbb{R}^+} u^2 \ln(u) e^{-u/2} \mathrm{d}u}_{(43)} - \underbrace{\int_{\mathbb{R}^+} \mathrm{Ei} \left(-\frac{u}{2} \right) u^2 e^{-u/2} \mathrm{d}u}_{(24)} - \frac{1}{2} (\ln(2) - \gamma) \underbrace{\int_{\mathbb{R}^+} u^2 e^{-u/2} \mathrm{d}u}_{=16} \quad (76)$$

$$= 12 + 8(\ln(2) - \gamma) - \underbrace{\left[16\mathrm{Ei}(-u) - 2(u^2 + 4u + 8)e^{-u/2}\mathrm{Ei} \left(-\frac{u}{2} \right) - 2(u + 5)e^{-u} \right]_0^{+\infty}}_{(22)} - 8(\ln(2) - \gamma) \quad (77)$$

$$= 12 + \lim_{\epsilon \rightarrow 0} [16(\ln(\epsilon) + \gamma) - 16(\ln(\epsilon - \ln(2)) + \gamma) - 10] = 2 + 16 \ln(2) \quad (78)$$

Hence the result,

$$\boxed{\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} u^2 \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v = 2 + 16 \ln(2) \approx 13.090355} \quad (79)$$

5 Evaluation of integral (4)

$$\begin{aligned} & \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} uv \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v \\ &= \int_{u=0}^{+\infty} up(u) \left(\underbrace{- \int_{v=0}^u v \ln \left(\frac{v}{u} \right) p(v) \mathrm{d}v}_{(a)} + \underbrace{\int_{v=u}^{+\infty} v \ln \left(\frac{v}{u} \right) p(v) \mathrm{d}v}_{(b)} \right) \mathrm{d}u \end{aligned} \quad (80)$$

For (a) of (80): using the integration by parts method,

$$\int_0^u v \ln\left(\frac{v}{u}\right) p(v) dv = \frac{1}{2} \left(\int_0^u v \ln(v) e^{-v/2} dv - \ln(u) \int_0^u v e^{-v/2} dv \right) \quad (81)$$

$$= \frac{1}{2} \left(\left(-2 [v \ln(v) e^{-v/2}]_0^u + 2 \underbrace{\int_0^u \ln(v) e^{-v/2} dv}_{(21)} + 2 \int_0^u e^{-v/2} dv \right) - \ln(u) \left(-2 [v e^{-v/2}]_0^u + 2 \int_0^u e^{-v/2} dv \right) \right) \quad (82)$$

$$= -u \ln(u) e^{-u/2} + \underbrace{\left[2 \text{Ei}\left(-\frac{v}{2}\right) - 2 e^{-v/2} \ln(v) \right]_0^u}_{(22)} - 2 (e^{-u/2} - 1) - \ln(u) (- (u e^{-u/2}) - 2 (e^{-u/2} - 1)) \quad (83)$$

$$= -u \ln(u) e^{-u/2} + 2 \text{Ei}\left(-\frac{u}{2}\right) - 2 e^{-u/2} \ln(u) - \lim_{\epsilon \rightarrow 0} (2 (\ln(\epsilon) - \ln(2) + \gamma) - 2 \ln(\epsilon)) - 2 e^{-u/2} + 2 - \ln(u) (- (u e^{-u/2}) - 2 (e^{-u/2} - 1)) \quad (84)$$

$$= -u \ln(u) e^{-u/2} + 2 \text{Ei}\left(-\frac{u}{2}\right) - 2 e^{-u/2} \ln(u) + 2 (\ln(2) - \gamma) - 2 e^{-u/2} + 2 + u \ln(u) e^{-u/2} + 2 \ln(u) e^{-u/2} - 2 \ln(u) \quad (85)$$

$$= 2 \left(\text{Ei}\left(-\frac{u}{2}\right) - e^{-u/2} - \ln(u) + (\ln(2) - \gamma) + 1 \right) \quad (86)$$

For (b) of (80):

$$\int_u^{+\infty} v \ln\left(\frac{v}{u}\right) p(v) dv = \frac{1}{2} \left(\int_u^{+\infty} v \ln(v) e^{-v/2} dv - \ln(u) \int_u^{+\infty} v e^{-v/2} dv \right) \quad (87)$$

$$= \frac{1}{2} \left(\left(-2 [v \ln(v) e^{-v/2}]_u^{+\infty} + 2 \underbrace{\int_u^{+\infty} \ln(v) e^{-v/2} dv}_{(21)} + 2 \int_u^{+\infty} e^{-v/2} dv \right) - \ln(u) \left(-2 [v e^{-v/2}]_u^{+\infty} + 2 \int_u^{+\infty} e^{-v/2} dv \right) \right) \quad (88)$$

$$= u \ln(u) e^{-u/2} + \left[2 \text{Ei}\left(-\frac{v}{2}\right) - 2 e^{-v/2} \ln(v) \right]_u^{+\infty} + 2 e^{-u/2} - \ln(u) (u e^{-u/2} + 2 e^{-u/2}) \quad (89)$$

$$= u \ln(u) e^{-u/2} - 2 \text{Ei}\left(-\frac{u}{2}\right) + 2 e^{-u/2} \ln(u) + 2 e^{-u/2} - u \ln(u) e^{-u/2} - 2 \ln(u) e^{-u/2} \quad (90)$$

$$= -2 \left(\text{Ei}\left(-\frac{u}{2}\right) - e^{-u/2} \right) \quad (91)$$

Gathering (a) and (b), one gets

$$\begin{aligned} & \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} uv \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v \\ &= \int_0^{+\infty} u e^{-u/2} \left(\ln(u) - 2\mathrm{Ei} \left(-\frac{u}{2} \right) - (\ln(2) - \gamma) + 2e^{-u/2} - 1 \right) \mathrm{d}u \end{aligned} \quad (92)$$

$$\begin{aligned} &= \underbrace{\int_0^{+\infty} u e^{-u/2} \ln(u) \mathrm{d}u}_{(36)} - 2 \underbrace{\int_0^{+\infty} u e^{-u/2} \mathrm{Ei} \left(-\frac{u}{2} \right) \mathrm{d}u}_{(23)} \\ &\quad - (\ln(2) - \gamma + 1) \underbrace{\int_0^{+\infty} u e^{-u/2} \mathrm{d}u}_{=4} + 2 \underbrace{\int_0^{+\infty} u e^{-u} \mathrm{d}u}_{=1} \end{aligned} \quad (93)$$

$$\begin{aligned} &= 4 + 4(\ln(2) - \gamma) - 2 \left[4\mathrm{Ei}(-u) - 2(u+2)\mathrm{Ei} \left(-\frac{u}{2} \right) e^{-u/2} - 2e^{-u} \right]_0^{+\infty} \\ &\quad - 4(\ln(2) - \gamma) - 4 + 2 \end{aligned} \quad (94)$$

$$= 2 + 2 \lim_{\epsilon \rightarrow 0} (4(\ln(\epsilon) + \gamma) - 4(\ln(\epsilon) - \ln(2) + \gamma) - 2) \quad (95)$$

$$= 8 \ln(2) - 2 \quad (96)$$

Hence the result,

$$\boxed{\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} uv \left| \ln \left(\frac{u}{v} \right) \right| p(u)p(v) \mathrm{d}u \mathrm{d}v = 8 \ln(2) - 2 \approx 3.545177} \quad (97)$$

6 Evaluation of integral (5)

$$\begin{aligned} & \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} \int_{w=0}^{+\infty} uw \left| \ln \left(\frac{u}{v} \right) \right| \left| \ln \left(\frac{w}{u} \right) \right| p(u)p(v)p(w) \mathrm{d}u \mathrm{d}v \mathrm{d}w \\ &= \int_{u=0}^{+\infty} u p(u) \underbrace{\left(\int_{v=0}^{+\infty} \left| \ln \left(\frac{u}{v} \right) \right| p(v) \mathrm{d}v \right)}_{\text{same as in (52)}} \underbrace{\left(\int_{w=0}^u w \left| \ln \left(\frac{w}{u} \right) \right| p(w) \mathrm{d}w \right)}_{\text{same as in (80)}} \mathrm{d}u \end{aligned} \quad (98)$$

$$\begin{aligned}
& \int_{u=0}^{+\infty} \int_{v=0}^{+\infty} \int_{w=0}^{+\infty} uv \left| \ln \left(\frac{u}{v} \right) \right| \left| \ln \left(\frac{w}{u} \right) \right| p(u)p(v)p(w) \mathrm{d}u \mathrm{d}v \mathrm{d}w \\
&= \int_0^{+\infty} u e^{-u/2} \left(\ln(u) - 2\mathrm{Ei} \left(-\frac{u}{2} \right) - (\ln(2) - \gamma) \right) \\
&\quad \times \left(\ln(u) - 2\mathrm{Ei} \left(-\frac{u}{2} \right) - (\ln(2) - \gamma) + 2e^{-u/2} - 1 \right) \mathrm{d}u \tag{99} \\
&= \underbrace{\int_0^{+\infty} u \ln(u)^2 e^{-u/2} \mathrm{d}u}_{\text{int. by parts}} + \underbrace{4 \int_0^{+\infty} u \mathrm{Ei} \left(-\frac{u}{2} \right)^2 e^{-u/2} \mathrm{d}u}_{(27)} + \underbrace{(\ln(2) - \gamma)^2 \int_0^{+\infty} u e^{-u/2} \mathrm{d}u}_{=4} \\
&\quad - 4 \underbrace{\int_0^{+\infty} u \ln(u) \mathrm{Ei} \left(-\frac{u}{2} \right) e^{-u/2} \mathrm{d}u}_{(26)} + 4(\ln(2) - \gamma) \underbrace{\int_0^{+\infty} u \mathrm{Ei} \left(-\frac{u}{2} \right) e^{-u/2} \mathrm{d}u}_{(23)} \\
&\quad - 2(\ln(2) - \gamma) \underbrace{\int_0^{+\infty} u \ln(u) e^{-u/2} \mathrm{d}u}_{(36)} \\
&\quad + 2 \underbrace{\int_0^{+\infty} u e^{-u} \ln(u) \mathrm{d}u}_{\text{as in (36)}} - 4 \underbrace{\int_0^{+\infty} u e^{-u} \mathrm{Ei} \left(-\frac{u}{2} \right) \mathrm{d}u}_{(25)} - 2(\ln(2) - \gamma) \underbrace{\int_0^{+\infty} u e^{-u} \mathrm{d}u}_{=1} \\
&\quad - \underbrace{\int_0^{+\infty} u e^{-u/2} \ln(u) \mathrm{d}u}_{(36)} + 2 \underbrace{\int_0^{+\infty} u e^{-u/2} \mathrm{Ei} \left(-\frac{u}{2} \right) \mathrm{d}u}_{(23)} + (\ln(2) - \gamma) \underbrace{\int_0^{+\infty} u e^{-u/2} \mathrm{d}u}_{=4} \tag{100}
\end{aligned}$$

$$\begin{aligned}
&= \left[-2(u+2)e^{-u/2} \ln(u)^2 \right]_0^{+\infty} + 4 \underbrace{\int_0^{+\infty} \ln(u) e^{-u/2} \mathrm{d}u}_{(32)} + 8 \underbrace{\int_0^{+\infty} \ln(u) \frac{e^{-u/2}}{u} \mathrm{d}u}_{\text{int. by parts}} \\
&\quad + 4 \left(-4 \ln(3)^2 - 4 \ln(3) + 8 \ln(2) \ln(3) - 8 \mathrm{Li}_2 \left(\frac{1}{3} \right) + \frac{2\pi^2}{3} \right) + 4(\ln(2) - \gamma)^2 \\
&\quad - 4 \left(-2 \ln(2)^2 + 4(\gamma - 1) \ln(2) - 2\gamma + \frac{\pi^2}{3} \right) \\
&\quad + 4(\ln(2) - \gamma) \left[4\mathrm{Ei}(-u) - 2(u+2)e^{-u/2} \mathrm{Ei} \left(-\frac{u}{2} \right) - 2e^{-u} \right]_0^{+\infty} \\
&\quad - 8(\ln(2) - \gamma) - 8(\ln(2) - \gamma)^2 \\
&\quad + 2(1 - \gamma) - 4 \left[\mathrm{Ei} \left(-\frac{3u}{2} \right) - (u+1)e^{-u} \mathrm{Ei} \left(-\frac{u}{2} \right) - \frac{2}{3} e^{-3u/2} \right]_0^{+\infty} - 2(\ln(2) - \gamma) \\
&\quad - (4 + 4(\ln(2) - \gamma)) + 2 \left[4\mathrm{Ei}(-u) - 2(u+2)e^{-u/2} \mathrm{Ei} \left(-\frac{u}{2} \right) - 2e^{-u} \right]_0^{+\infty} + 4(\ln(2) - \gamma) \tag{101}
\end{aligned}$$

$$\begin{aligned}
&= 4 \lim_{\epsilon \rightarrow 0} \ln(\epsilon)^2 + 8(\ln(2) - \gamma) + 8 \left(\left[\ln(u) \text{Ei} \left(-\frac{u}{2} \right) \right]_0^{+\infty} - \underbrace{\int_0^{+\infty} \frac{1}{u} \text{Ei} \left(-\frac{u}{2} \right) du}_{(29)} \right) \\
&\quad - 16 \ln(3)^2 - 16 \ln(3) + 32 \ln(2) \ln(3) - 32 \text{Li}_2 \left(\frac{1}{3} \right) + \frac{8\pi^2}{3} + 4(\ln(2) - \gamma)^2 \\
&\quad + 8 \ln(2)^2 - 16(\gamma - 1) \ln(2) + 8\gamma - \frac{4\pi^2}{3} \\
&\quad - 4 \lim_{\epsilon \rightarrow 0} (\ln(2) - \gamma) (4(\ln(\epsilon) + \gamma) - 4(\ln(\epsilon) - \ln(2) + \gamma) - 2) \\
&\quad - 8(\ln(2) - \gamma) - 8(\ln(2) - \gamma)^2 \\
&\quad + 2 - 2\gamma + 4 \lim_{\epsilon \rightarrow 0} \left(\ln(\epsilon) + \ln(3) - \ln(2) + \gamma - (\ln(\epsilon) - \ln(2) + \gamma) - \frac{2}{3} \right) - 2(\ln(2) - \gamma) \\
&\quad - 4 - 4(\ln(2) - \gamma) - 2 \lim_{\epsilon \rightarrow 0} (4(\ln(\epsilon) + \gamma) - 4(\ln(\epsilon) - \ln(2) + \gamma) - 2) + 4(\ln(2) - \gamma)
\end{aligned} \tag{102}$$

$$\begin{aligned}
&= \lim_{\epsilon \rightarrow 0} \left(4 \ln(\epsilon)^2 - 8 \ln(\epsilon)^2 + 8(\ln(2) - \gamma) \ln(\epsilon) + 4 \ln(\epsilon)^2 - 8(\ln(2) - \gamma) \ln(\epsilon) + 4(\ln(2) - \gamma)^2 + \frac{2\pi^2}{3} \right) \\
&\quad - 16 \ln(3)^2 - 16 \ln(3) + 32 \ln(2) \ln(3) - 32 \text{Li}_2 \left(\frac{1}{3} \right) + \frac{8\pi^2}{3} \\
&\quad + 8 \ln(2)^2 - 16(\gamma - 1) \ln(2) + 8\gamma - \frac{4\pi^2}{3} \\
&\quad - 4(\ln(2) - \gamma) (4 \ln(2) - 2) - 4(\ln(2) - \gamma)^2 \\
&\quad + 2 - 2\gamma + 4 \left(\ln(3) - \frac{2}{3} \right) - 2(\ln(2) - \gamma) \\
&\quad - 4 - 4(\ln(2) - \gamma) - 2(4 \ln(2) - 2) + 4(\ln(2) - \gamma)
\end{aligned} \tag{103}$$

$$= 2\pi^2 - 32 \text{Li}_2 \left(\frac{1}{3} \right) - 16 \ln(3)^2 - 8 \ln(2)^2 + 32 \ln(2) \ln(3) - 12 \ln(3) + 14 \ln(2) - \frac{2}{3} \tag{104}$$

Hence the result,

$$\boxed{\int_{u=0}^{+\infty} \int_{v=0}^{+\infty} \int_{w=0}^{+\infty} u w \left| \ln \left(\frac{u}{v} \right) \right| \left| \ln \left(\frac{w}{u} \right) \right| p(u) p(v) p(w) du dv dw \approx 5.087625} \tag{105}$$

References

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