Technical Note
Details on SIGMA integrals evaluation

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The aim of this note is to provide details on the evaluation of the following integrals:

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v, \tag{1}$$

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} \left(u\ln\left(\frac{u}{v}\right)\right)^2 p(u)p(v)\mathrm{d}u\mathrm{d}v, \tag{2}$$

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u^2\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v, \tag{3}$$

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} uv\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v, \tag{4}$$

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty}\int_{w=0}^{+\infty} uw\left|\ln\left(\frac{u}{v}\right)\right|\left|\ln\left(\frac{w}{u}\right)\right| p(u)p(v)p(w)\mathrm{d}u\mathrm{d}v\mathrm{d}w, \tag{5}$$

where $p(\cdot)$ refers to the probability density function of a χ^2 distributed random variable.

$$p(x) = \frac{1}{2}\exp\left(-\frac{x}{2}\right). \tag{6}$$

1 Preliminary results

In order to evaluate (1)–(5), a number of intermediate integrals need to be considered.

1.1 Euler-Mascheroni constant

A first integral is the the following:

$$\begin{aligned}
\int_{\mathbb{R}^+} \ln(x)e^{-x}\mathrm{d}x &= \lim_{\mu\to 0}\frac{\mathrm{d}}{\mathrm{d}\mu}\int_{\mathbb{R}^+} x^{\mu}e^{-x}\mathrm{d}x &(7)\\
&= \lim_{\mu\to 0}\frac{\mathrm{d}}{\mathrm{d}\mu}\Gamma(\mu+1) &(8)\\
&= \lim_{\mu\to 0}\Gamma(\mu+1)\psi(\mu+1) &(9)\\
&= \Gamma(1)\psi^{(0)}(1) = -\gamma, &(10)
\end{aligned}$$

where $\psi^{(n)}(x)$ est the polygamma function, defined as the $n+1$-th derivative of $\ln(\Gamma(x))$ [1, Sec. 6.3] and $\gamma = 0.577215...$ is the Euler-Mascheroni constant. Similarly,

$$\int_{\mathbb{R}^+} \ln(x)^2 e^{-x} \mathrm{d}x = \lim_{\mu \to 0} \frac{\mathrm{d}^2}{\mathrm{d}\mu^2} \int_{\mathbb{R}^+} x^\mu e^{-x} \mathrm{d}x \tag{11}$$

$$= \lim_{\mu \to 0} \frac{\mathrm{d}^2}{\mathrm{d}\mu^2} \Gamma(\mu+1) \tag{12}$$

$$= \lim_{\mu \to 0} \Gamma(\mu+1) \left((\psi^{(0)}(\mu+1))^2 + \psi^{(1)}(\mu+1) \right) \tag{13}$$

$$= \Gamma(1) \left((\psi^{(0)}(1))^2 + \psi^{(1)}(1) \right) = \gamma^2 + \frac{\pi^2}{6}. \tag{14}$$

1.2 Exponential integral Ei

In the following computation the exponential integral $\mathrm{Ei}(\cdot)$ is often used. It is defined as

$$\mathrm{Ei}\,(x) = -\int_{-x}^{+\infty} \frac{e^{-x}}{x} \mathrm{d}x. \tag{15}$$

An important property of this integral is its expansion around 0, for $x > 0$:

$$\mathrm{Ei}\left(-\frac{x}{2}\right) = \ln(x) - \ln(2) + \gamma + O(x). \tag{16}$$

Besides, it appears in integrals involving $\ln$ and $\exp$ functions.

$$\int_a^b \ln(x) e^{-x/2} \mathrm{d}x = \left[-2e^{-x/2}\ln(x)\right]_a^b - \int_a^b -2\frac{e^{-x/2}}{x}\mathrm{d}x \tag{17}$$

$$= \left[-2e^{-x/2}\ln(x)\right]_a^b - \int_{a/2}^{b/2} -2\frac{e^{-y}}{y}\mathrm{d}y \tag{18}$$

$$= \left[-2e^{-x/2}\ln(x)\right]_a^b + 2\left(\int_{a/2}^{+\infty} \frac{e^{-y}}{y}\mathrm{d}y - \int_{b/2}^{+\infty} \frac{e^{-y}}{y}\mathrm{d}y\right) \tag{19}$$

$$= \left[-2e^{-x/2}\ln(x)\right]_a^b + 2\left(-\mathrm{Ei}\left(-\frac{a}{2}\right) + \mathrm{Ei}\left(-\frac{b}{2}\right)\right) \tag{20}$$

$$= \left[2\mathrm{Ei}\left(-\frac{x}{2}\right) - 2e^{-x/2}\ln(x)\right]_a^b. \tag{21}$$

Thanks to (16), a special case for $a = 0$ yields:

$$\lim_{a \to 0^+} 2\mathrm{Ei}\left(-\frac{x}{2}\right) - 2e^{-x/2}\ln(x) = \lim_{a \to 0^+} 2\ln(x) - 2\ln(2) + 2\gamma - 2\ln(x) = -2(\ln(2) - \gamma) \tag{22}$$

The following expressions are obtained from [2, Sec. 1.3.2, eq (1) and (3)].

$$\int x e^{-x/2} \mathrm{Ei}\left(-\frac{x}{2}\right) \mathrm{d}x = 4\mathrm{Ei}\,(-x) - 2(x+2)e^{-x/2}\mathrm{Ei}\left(-\frac{x}{2}\right) - 2e^{-x}, \tag{23}$$

$$\int x^2 e^{-x/2} \mathrm{Ei}\left(-\frac{x}{2}\right) \mathrm{d}x = 16\mathrm{Ei}\,(-x) - 2(x^2+4x+8)e^{-x/2}\mathrm{Ei}\left(-\frac{x}{2}\right) - 2(x+5)e^{-x}, \tag{24}$$

$$\int xe^{-x}\mathrm{Ei}\left(-\frac{x}{2}\right)\mathrm{d}x = \mathrm{Ei}\left(-\frac{3x}{2}\right) - (x+1)e^{-x}\mathrm{Ei}\left(-\frac{x}{2}\right) - \frac{2}{3}e^{-3x/2}, \tag{25}$$

from [2, Sec. 2.5.10 eq (6)]

$$\int_0^{+\infty} x\ln(x)e^{-x/2}\mathrm{Ei}\left(-\frac{x}{2}\right)\mathrm{d}x = -2\ln(2)^2 + 4(\gamma-1)\ln(2) - 2\gamma + \frac{\pi^2}{3}, \tag{26}$$

from [2, Sec. 2.5.12 eq (8)]

$$\int_0^{+\infty} xe^{-x/2}\mathrm{Ei}\left(-\frac{x}{2}\right)^2\mathrm{d}x = -4\ln(3)^2 - 4\ln(3) + 8\ln(2)\ln(3) - 8\mathrm{Li}_2\left(\frac{1}{3}\right) + \frac{2\pi^2}{3}, \tag{27}$$

where $\mathrm{Li}_2(x)$ denotes the polylogarithm, defined as

$$\mathrm{Li}_n(x) = \sum_{k=1}^{+\infty}\frac{x}{k^n}. \tag{28}$$

A numerical evaluation of the specific value of interest yields: $\mathrm{Li}_2\left(\frac{1}{3}\right) \approx 0.366213$.

Finally, from [2, Sec. 1.3.1 eq (11)]

$$\begin{aligned}\int_0^{+\infty}\frac{1}{x}\mathrm{Ei}\left(-\frac{x}{2}\right)^2\mathrm{d}x &= \lim_{\epsilon\to 0} -\frac{1}{2}\left((\ln(\epsilon)-\ln(2)+\gamma)^2+\zeta(2)\right) - \sum_{k=1}^{+\infty}\frac{(-1)^k\epsilon^k}{k!k^2 2^k} \\ &= \lim_{\epsilon\to 0} -\frac{1}{2}\left((\ln(\epsilon)^2 - 2(\ln(2)-\gamma)\ln(\epsilon) + (\ln(2)-\gamma)^2 + \frac{\pi^2}{6}\right).\end{aligned} \tag{29}$$

1.3 Intermediate integrals

Using a simple change of variable,

$$\int_{\mathbb{R}^+}\ln(u)e^{-u/2}\mathrm{d}u = 2\ln(2)\int_{\mathbb{R}^+}e^{-x}\mathrm{d}x + 2\underbrace{\int_{\mathbb{R}^+}\ln(x)e^{-x}\mathrm{d}x}_{(10)} \tag{30}$$

$$= 2\ln(2) - 2\gamma. \tag{31}$$

$$\boxed{\int_{\mathbb{R}^+}\ln(u)e^{-u/2}\mathrm{d}u = 2(\ln(2)-\gamma)} \tag{32}$$

Using the integration by parts method,

$$\int_{\mathbb{R}^+}u\ln(u)e^{-u/2}\mathrm{d}u = \left[u\ln(u)(-2e^{-u/2})\right]_0^{+\infty} - \int_{\mathbb{R}^+}(\ln(u)+1)(-2e^{-u/2})\mathrm{d}u \tag{33}$$

$$= 2\underbrace{\int_{\mathbb{R}^+}\ln(u)e^{-u/2}\mathrm{d}u}_{(32)} + 2\underbrace{\int_{\mathbb{R}^+}e^{-u/2}\mathrm{d}u}_{=2} \tag{34}$$

$$= 4(\ln(2)-\gamma) + 4. \tag{35}$$

$$\boxed{\int_{\mathbb{R}^+} u\ln(u)e^{-u/2}\mathrm{d}u = 4 + 4(\ln(2) - \gamma)} \tag{36}$$

Again, using a change of variable,

$$\int_{\mathbb{R}^+} \ln(u)^2 e^{-u/2}\mathrm{d}u = 2\ln(2)^2 \int_{\mathbb{R}^+} e^{-x}\mathrm{d}x + 4\ln(2) \underbrace{\int_{\mathbb{R}^+} \ln(x)e^{-x}\mathrm{d}x}_{(10)} + 2 \underbrace{\int_{\mathbb{R}^+} \ln(x)^2 e^{-x}\mathrm{d}x}_{(14)} \tag{37}$$

$$= 2(\ln(2))^2 - 4\ln(2)\gamma + 2\left(\gamma^2 + \frac{\pi^2}{6}\right). \tag{38}$$

$$\boxed{\int_{\mathbb{R}^+} \ln(u)^2 e^{-u/2}\mathrm{d}u = 2(\ln(2) - \gamma)^2 + \frac{\pi^2}{3}} \tag{39}$$

By using the integration by parts method,

$$\int_{\mathbb{R}^+} u^2\ln(u)e^{-u/2}\mathrm{d}u = \left[u^2\ln(u)(-2e^{-u/2})\right]_0^{+\infty} - \int_{\mathbb{R}^+} (2u\ln(u) + u)(-2e^{-u/2})\mathrm{d}u \tag{40}$$

$$= 4\underbrace{\int_{\mathbb{R}^+} u\ln(u)e^{-u/2}\mathrm{d}u}_{(36)} + 2\underbrace{\int_{\mathbb{R}^+} ue^{-u/2}\mathrm{d}u}_{=4} \tag{41}$$

$$= 16 + 16(\ln(2) - \gamma) + 8. \tag{42}$$

$$\boxed{\int_{\mathbb{R}^+} u^2\ln(u)e^{-u/2}\mathrm{d}u = 24 + 16(\ln(2) - \gamma)} \tag{43}$$

$$\int_{\mathbb{R}^+} u\ln(u)^2 e^{-u/2}\mathrm{d}u = \left[u\ln(u)^2(-2e^{-u/2})\right]_0^{+\infty} - \int_{\mathbb{R}^+} (\ln(u)^2 + 2\ln(u))(-2e^{-u/2})\mathrm{d}u \tag{44}$$

$$= 2\underbrace{\int_{\mathbb{R}^+} \ln(u)^2 e^{-u/2}\mathrm{d}u}_{(39)} + 4\underbrace{\int_{\mathbb{R}^+} \ln(u)e^{-u/2}\mathrm{d}u}_{(32)} \tag{45}$$

$$= 4(\ln(2) - \gamma)^2 + \frac{2\pi^2}{3} + 8(\ln(2) - \gamma). \tag{46}$$

$$\boxed{\int_{\mathbb{R}^+} u\ln(u)^2 e^{-u/2}\mathrm{d}u = 4(\ln(2) - \gamma)^2 + 8(\ln(2) - \gamma) + \frac{2\pi^2}{3}} \tag{47}$$

$$\int_{\mathbb{R}^+} u^2\ln(u)^2 e^{-u/2}\mathrm{d}u = \left[u^2\ln(u)^2(-2e^{-u/2})\right]_0^{+\infty} - \int_{\mathbb{R}^+} (2u\ln(u)^2 + 2u\ln(u))(-2e^{-u/2})\mathrm{d}u \tag{48}$$

$$= 4\underbrace{\int_{\mathbb{R}^+} u\ln(u)^2 e^{-u/2}\mathrm{d}u}_{(47)} + 4\underbrace{\int_{\mathbb{R}^+} u\ln(u)e^{-u/2}\mathrm{d}u}_{(36)} \tag{49}$$

$$= 16(\ln(2) - \gamma)^2 + 32(\ln(2) - \gamma) + \frac{8\pi^2}{3} + 16 + 16(\ln(2) - \gamma). \tag{50}$$

$$\boxed{\int_{\mathbb{R}^+} u^2\ln(u)^2 e^{-u/2}\mathrm{d}u = 16(\ln(2) - \gamma)^2 + 48(\ln(2) - \gamma) + 16 + \frac{8\pi^2}{3}} \tag{51}$$

2 Evaluation of integral (1)

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v$$
$$= \int_{u=0}^{+\infty} up(u)\left(\underbrace{\int_{v=0}^{u}\ln\left(\frac{u}{v}\right)p(v)\mathrm{d}v}_{(\mathbf{a})} - \underbrace{\int_{v=u}^{+\infty}\ln\left(\frac{u}{v}\right)p(v)\mathrm{d}v}_{(\mathbf{b})}\right)\mathrm{d}u \tag{52}$$

For $(\mathbf{a})$ **of (52):** using (6),

$$\int_{v=0}^{u}\ln\left(\frac{u}{v}\right)p(v)\mathrm{d}v = \frac{1}{2}\left(\ln(u)\int_{v=0}^{u} e^{-v/2}\mathrm{d}v - \underbrace{\int_{v=0}^{u}\ln(v)e^{-v/2}\mathrm{d}v}_{(21)}\right) \tag{53}$$
$$= \frac{1}{2}\left(2\ln(u)\left(1-e^{-u/2}\right) - \underbrace{\left[2\mathrm{Ei}\left(-\frac{v}{2}\right) - 2\ln(v)e^{-v/2}\right]_{0}^{u}}_{(22)}\right) \tag{54}$$
$$= \ln(u) - \ln(u)e^{-u/2} - \left(\mathrm{Ei}\left(-\frac{u}{2}\right) - \ln(u)e^{-u/2} + (\ln(2)-\gamma)\right) \tag{55}$$
$$= \ln(u) - \mathrm{Ei}\left(-\frac{u}{2}\right) - (\ln(2)-\gamma). \tag{56}$$

where $\mathrm{Ei}(\cdot)$ is the Exponential integral [1, Sec. 5]

For $(\mathbf{b})$ **of (52):**

$$\int_{v=u}^{+\infty}\ln\left(\frac{u}{v}\right)p(v)\mathrm{d}v = \frac{1}{2}\left(\ln(u)\int_{v=u}^{+\infty} e^{-v/2}\mathrm{d}v - \underbrace{\int_{v=u}^{+\infty}\ln(v)e^{-v/2}\mathrm{d}v}_{(21)}\right) \tag{57}$$
$$= \frac{1}{2}\left(2\ln(u)e^{-u/2} - \left[2\mathrm{Ei}\left(-\frac{v}{2}\right) - 2\ln(v)e^{-v/2}\right]_{u}^{+\infty}\right) \tag{58}$$
$$= \ln(u)e^{-u/2} - \left(-\mathrm{Ei}\left(-\frac{u}{2}\right) + \ln(u)e^{-u/2}\right) \tag{59}$$
$$= \mathrm{Ei}\left(-\frac{u}{2}\right) \tag{60}$$

Gathering $(\mathbf{a})$ and $(\mathbf{b})$, one gets

$$
\begin{aligned}
&\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v \\
&\quad= \frac{1}{2}\int_{\mathbb{R}^+} ue^{-u/2}\left(\ln(u) - 2\mathrm{Ei}\left(-\frac{u}{2}\right) - (\ln(2)-\gamma)\right)\mathrm{d}u && (61)\\
&\quad= \frac{1}{2}\underbrace{\int_{\mathbb{R}^+} u\ln(u)e^{-u/2}\mathrm{d}u}_{(36)} - \underbrace{\int_{\mathbb{R}^+}\mathrm{Ei}\left(-\frac{u}{2}\right)ue^{-u/2}\mathrm{d}u}_{(23)} - \frac{1}{2}(\ln(2)-\gamma)\underbrace{\int_{\mathbb{R}^+} ue^{-u/2}\mathrm{d}u}_{=4} && (62)\\
&\quad= 2 + 2(\ln(2)-\gamma) - \underbrace{\left[4\mathrm{Ei}\left(-u\right) - 2(u+2)e^{-u/2}\mathrm{Ei}\left(-\frac{u}{2}\right) - 2e^{-u}\right]_0^{+\infty}}_{(16)} - 2(\ln(2)-\gamma) && (63)\\
&\quad= 2 - 2 + \lim_{\epsilon\to 0}\left(4\gamma + 4\ln(\epsilon) - 4\left(\ln(\epsilon) - \ln(2) + \gamma\right)\right) = 4\ln(2). && (64)
\end{aligned}
$$

Hence the result,

$$
\boxed{\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v = 4\ln(2) \approx 2.772589} \quad (65)
$$

3 Evaluation of integral (2)

$$
\begin{aligned}
&\int_{u=0}^{+\infty}\int_{v=0}^{+\infty}\left(u\ln\left(\frac{u}{v}\right)\right)^2 p(u)p(v)\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{4}\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u^2\left(\ln(u)^2 - 2\ln(u)\ln(v) + \ln(v)^2\right)e^{-u/2}e^{-v/2}\mathrm{d}u\mathrm{d}v \quad (66)\\
&= \frac{1}{4}\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u^2\ln(u)^2 e^{-u/2}e^{-v/2}\mathrm{d}u\mathrm{d}v - \frac{1}{2}\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u^2\ln(u)\ln(v)e^{-u/2}e^{-v/2}\mathrm{d}u\mathrm{d}v \\
&\quad + \frac{1}{4}\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u^2\ln(v)^2 e^{-u/2}e^{-v/2}\mathrm{d}u\mathrm{d}v \quad (67)\\
&= \frac{1}{4}\underbrace{\int_{u=0}^{+\infty} u^2\ln(u)^2 e^{-u/2}\mathrm{d}u}_{(51)}\underbrace{\int_{v=0}^{+\infty} e^{-v/2}\mathrm{d}v}_{=2} - \frac{1}{2}\underbrace{\int_{u=0}^{+\infty} u^2\ln(u)e^{-u/2}\mathrm{d}u}_{(43)}\underbrace{\int_{v=0}^{+\infty}\ln(v)e^{-v/2}\mathrm{d}v}_{(32)} \\
&\quad + \frac{1}{4}\underbrace{\int_{u=0}^{+\infty} u^2 e^{-u/2}\mathrm{d}u}_{=16}\underbrace{\int_{v=0}^{+\infty}\ln(v)^2 e^{-v/2}\mathrm{d}v}_{(39)} \quad (68)\\
&= \frac{1}{2}\left(16(\ln(2)-\gamma)^2 + 48(\ln(2)-\gamma) + 16 + \frac{8\pi^2}{3}\right) - \left(24 + 16(\ln(2)-\gamma)\right)(\ln(2)-\gamma) \\
&\quad + 8(\ln(2)-\gamma)^2 + \frac{4\pi^2}{3} \quad (69)\\
&= 8(\ln(2)-\gamma)^2 + 24(\ln(2)-\gamma) + 8 + \frac{4\pi^2}{3} - 24(\ln(2)-\gamma) - 16(\ln(2)-\gamma)^2 \\
&\quad + 8(\ln(2)-\gamma)^2 + \frac{4\pi^2}{3} \quad (70)\\
&= 8 + \frac{8\pi^2}{3} \quad (71)
\end{aligned}
$$

Hence the result,

$$\boxed{\int_{u=0}^{+\infty}\int_{v=0}^{+\infty}\left(u\ln\left(\frac{u}{v}\right)\right)^2 p(u)p(v)\mathrm{d}u\mathrm{d}v = 8 + \frac{8\pi^2}{3} \approx 34.318945} \quad (72)$$

4 Evaluation of integral (3)

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u^2 \left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v \tag{73}$$

$$= \int_{u=0}^{+\infty} u^2 p(u) \underbrace{\left(\int_{v=0}^{+\infty} \ln\left(\frac{u}{v}\right) p(v)\mathrm{d}v\right)}_{\text{same as in (52)}} \mathrm{d}u \tag{74}$$

$$= \frac{1}{2}\int_{\mathbb{R}+} u^2 e^{-u/2}\left(\ln(u) - 2\mathrm{Ei}\left(-\frac{u}{2}\right) - (\ln(2)-\gamma)\right)\mathrm{d}u \tag{75}$$

$$= \underbrace{\frac{1}{2}\int_{\mathbb{R}+} u^2\ln(u)e^{-u/2}\mathrm{d}u}_{(43)} - \underbrace{\int_{\mathbb{R}+}\mathrm{Ei}\left(-\frac{u}{2}\right)u^2 e^{-u/2}\mathrm{d}u}_{(24)} - \frac{1}{2}(\ln(2)-\gamma)\underbrace{\int_{\mathbb{R}+} u^2 e^{-u/2}\mathrm{d}u}_{=16} \tag{76}$$

$$= 12 + 8(\ln(2)-\gamma) - \underbrace{\left[16\mathrm{Ei}\left(-u\right) - 2(u^2+4u+8)e^{-u/2}\mathrm{Ei}\left(-\frac{u}{2}\right) - 2(u+5)e^{-u}\right]_0^{+\infty}}_{(22)}$$

$$- 8(\ln(2)-\gamma) \tag{77}$$

$$= 12 + \lim_{\epsilon\to 0}\left[16\left(\ln(\epsilon)+\gamma\right) - 16\left(\ln(\epsilon - \ln(2)+\gamma\right) - 10\right] = 2 + 16\ln(2) \tag{78}$$

Hence the result,

$$\boxed{\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} u^2\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v = 2+16\ln(2) \approx 13.090355} \tag{79}$$

5 Evaluation of integral (4)

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} uv\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v$$

$$= \int_{u=0}^{+\infty} up(u)\left(-\underbrace{\int_{v=0}^{u} v\ln\left(\frac{v}{u}\right)p(v)\mathrm{d}v}_{(\mathbf{a})} + \underbrace{\int_{v=u}^{+\infty} v\ln\left(\frac{v}{u}\right)p(v)\mathrm{d}w}_{(\mathbf{b})}\right)\mathrm{d}u \tag{80}$$

For $(\mathbf{a})$ **of** (80)**:** using the integration by parts method,

$$\int_0^u v\ln\left(\frac{v}{u}\right)p(v)\mathrm{d}v = \frac{1}{2}\left(\int_0^u v\ln(v)e^{-v/2}\mathrm{d}v - \ln(u)\int_0^u ve^{-v/2}\mathrm{d}v\right) \tag{81}$$

$$= \frac{1}{2}\left(\left(-2\left[v\ln(v)e^{-v/2}\right]_0^u + 2\underbrace{\int_0^u \ln(v)e^{-v/2}\mathrm{d}v}_{(21)} + 2\int_0^u e^{-v/2}\mathrm{d}v\right)\right.$$
$$\left. - \ln(u)\left(-2\left[ve^{-v/2}\right]_0^u + 2\int_0^u e^{-v/2}\mathrm{d}v\right)\right) \tag{82}$$

$$= -u\ln(u)e^{-u/2} + \underbrace{\left[2\mathrm{Ei}\left(-\frac{v}{2}\right) - 2e^{-v/2}\ln(v)\right]_0^u}_{(22)} - 2\left(e^{-u/2} - 1\right)$$
$$- \ln(u)\left(-\left(ue^{-u/2}\right) - 2\left(e^{-u/2} - 1\right)\right) \tag{83}$$

$$= -u\ln(u)e^{-u/2} + 2\mathrm{Ei}\left(-\frac{u}{2}\right) - 2e^{-u/2}\ln(u) - \lim_{\epsilon\to 0}\left(2\left(\ln(\epsilon) - \ln(2) + \gamma\right) - 2\ln(\epsilon)\right) - 2e^{-u/2} + 2$$
$$- \ln(u)\left(-\left(ue^{-u/2}\right) - 2\left(e^{-u/2} - 1\right)\right) \tag{84}$$

$$= -u\ln(u)e^{-u/2} + 2\mathrm{Ei}\left(-\frac{u}{2}\right) - 2e^{-u/2}\ln(u) + 2\left(\ln(2) - \gamma\right) - 2e^{-u/2} + 2$$
$$+ u\ln(u)e^{-u/2} + 2\ln(u)e^{-u/2} - 2\ln(u) \tag{85}$$

$$= 2\left(\mathrm{Ei}\left(-\frac{u}{2}\right) - e^{-u/2} - \ln(u) + \left(\ln(2) - \gamma\right) + 1\right) \tag{86}$$

For $(\mathbf{b})$ **of** (80)**:**

$$\int_u^{+\infty} v\ln\left(\frac{v}{u}\right)p(v)\mathrm{d}v = \frac{1}{2}\left(\int_u^{+\infty} v\ln(v)e^{-v/2}\mathrm{d}v - \ln(u)\int_u^{+\infty} ve^{-v/2}\mathrm{d}v\right) \tag{87}$$

$$= \frac{1}{2}\left(\left(-2\left[v\ln(v)e^{-v/2}\right]_u^{\infty} + 2\underbrace{\int_u^{+\infty} \ln(v)e^{-v/2}\mathrm{d}v}_{(21)} + 2\int_u^{+\infty} e^{-v/2}\mathrm{d}v\right)\right.$$
$$\left. - \ln(u)\left(-2\left[ve^{-v/2}\right]_u^{+\infty} + 2\int_u^{+\infty} e^{-v/2}\mathrm{d}v\right)\right) \tag{88}$$

$$= u\ln(u)e^{-u/2} + \left[2\mathrm{Ei}\left(-\frac{v}{2}\right) - 2e^{-v/2}\ln(v)\right]_u^{+\infty} + 2e^{-u/2} - \ln(u)\left(ue^{-u/2} + 2e^{-u/2}\right) \tag{89}$$

$$= u\ln(u)e^{-u/2} - 2\mathrm{Ei}\left(-\frac{u}{2}\right) + 2e^{-u/2}\ln(u) + 2e^{-u/2} - u\ln(u)e^{-u/2} - 2\ln(u)e^{-u/2} \tag{90}$$

$$= -2\left(\mathrm{Ei}\left(-\frac{u}{2}\right) - e^{-u/2}\right) \tag{91}$$

Gathering $(\mathbf{a})$ and $(\mathbf{b})$, one gets

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} uv\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v$$

$$= \int_{0}^{+\infty} ue^{-u/2}\left(\ln(u) - 2\mathrm{Ei}\left(-\frac{u}{2}\right) - (\ln(2)-\gamma) + 2e^{-u/2} - 1\right)\mathrm{d}u \tag{92}$$

$$= \underbrace{\int_{0}^{+\infty} ue^{-u/2}\ln(u)\mathrm{d}u}_{(36)} - 2\underbrace{\int_{0}^{+\infty} ue^{-u/2}\mathrm{Ei}\left(-\frac{u}{2}\right)\mathrm{d}u}_{(23)}$$

$$- (\ln(2)-\gamma+1)\underbrace{\int_{0}^{+\infty} ue^{-u/2}\mathrm{d}u}_{=4} + 2\underbrace{\int_{0}^{+\infty} ue^{-u}\mathrm{d}u}_{=1} \tag{93}$$

$$= 4 + 4(\ln(2)-\gamma) - 2\left[4\mathrm{Ei}\left(-u\right) - 2(u+2)\mathrm{Ei}\left(-\frac{u}{2}\right)e^{-u/2} - 2e^{-u}\right]_0^{+\infty}$$
$$- 4(\ln(2)-\gamma) - 4 + 2 \tag{94}$$

$$= 2 + 2\lim_{\epsilon\to 0}\left(4\left(\ln(\epsilon)+\gamma\right) - 4\left(\ln(\epsilon) - \ln(2) + \gamma\right) - 2\right) \tag{95}$$

$$= 8\ln(2) - 2 \tag{96}$$

Hence the result,

$$\boxed{\int_{u=0}^{+\infty}\int_{v=0}^{+\infty} uv\left|\ln\left(\frac{u}{v}\right)\right| p(u)p(v)\mathrm{d}u\mathrm{d}v = 8\ln(2) - 2 \approx 3.545177} \tag{97}$$

6 Evaluation of integral (5)

$$\int_{u=0}^{+\infty}\int_{v=0}^{+\infty}\int_{w=0}^{+\infty} uw\left|\ln\left(\frac{u}{v}\right)\right|\left|\ln\left(\frac{w}{u}\right)\right| p(u)p(v)p(w)\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

$$= \int_{u=0}^{+\infty} up(u)\underbrace{\left(\int_{v=0}^{+\infty}\left|\ln\left(\frac{u}{v}\right)\right| p(v)\mathrm{d}v\right)}_{\text{same as in (52)}}\underbrace{\left(\int_{w=0}^{u} w\left|\ln\left(\frac{w}{u}\right)\right| p(w)\mathrm{d}w\right)}_{\text{same as in (80)}}\mathrm{d}u \tag{98}$$

$$
\begin{aligned}
&\int_{u=0}^{+\infty}\int_{v=0}^{+\infty}\int_{w=0}^{+\infty} uw\left|\ln\left(\frac{u}{v}\right)\right|\left|\ln\left(\frac{w}{u}\right)\right| p(u)p(v)p(w)\mathrm{d}u\mathrm{d}v\mathrm{d}w\\
&\quad=\int_0^{+\infty} ue^{-u/2}\left(\ln(u)-2\mathrm{Ei}\left(-\frac{u}{2}\right)-(\ln(2)-\gamma)\right)\\
&\qquad\times\left(\ln(u)-2\mathrm{Ei}\left(-\frac{u}{2}\right)-(\ln(2)-\gamma)+2e^{-u/2}-1\right)\mathrm{d}u \qquad (99)\\
&\quad=\underbrace{\int_0^{+\infty} u\ln(u)^2 e^{-u/2}\mathrm{d}u}_{\text{int. by parts}}+4\underbrace{\int_0^{+\infty} u\mathrm{Ei}\left(-\frac{u}{2}\right)^2 e^{-u/2}\mathrm{d}u}_{(27)}+(\ln(2)-\gamma)^2\underbrace{\int_0^{+\infty} ue^{-u/2}\mathrm{d}u}_{=4}\\
&\qquad-4\underbrace{\int_0^{+\infty} u\ln(u)\mathrm{Ei}\left(-\frac{u}{2}\right)e^{-u/2}\mathrm{d}u}_{(26)}+4(\ln(2)-\gamma)\underbrace{\int_0^{+\infty} u\mathrm{Ei}\left(-\frac{u}{2}\right)e^{-u/2}\mathrm{d}u}_{(23)}\\
&\qquad-2(\ln(2)-\gamma)\underbrace{\int_0^{+\infty} u\ln(u)e^{-u/2}\mathrm{d}u}_{(36)}\\
&\qquad+2\underbrace{\int_0^{+\infty} ue^{-u}\ln(u)\mathrm{d}u}_{\text{as in (36)}}-4\underbrace{\int_0^{+\infty} ue^{-u}\mathrm{Ei}\left(-\frac{u}{2}\right)\mathrm{d}u}_{(25)}-2(\ln(2)-\gamma)\underbrace{\int_0^{+\infty} ue^{-u}\mathrm{d}u}_{=1}\\
&\qquad-\underbrace{\int_0^{+\infty} ue^{-u/2}\ln(u)\mathrm{d}u}_{(36)}+2\underbrace{\int_0^{+\infty} ue^{-u/2}\mathrm{Ei}\left(-\frac{u}{2}\right)\mathrm{d}u}_{(23)}+(\ln(2)-\gamma)\underbrace{\int_0^{+\infty} ue^{-u/2}\mathrm{d}u}_{=4} \qquad (100)
\end{aligned}
$$

$$
\begin{aligned}
&=\left[-2(u+2)e^{-u/2}\ln(u)^2\right]_0^{+\infty}+4\underbrace{\int_0^{+\infty}\ln(u)e^{-u/2}\mathrm{d}u}_{(32)}+8\underbrace{\int_0^{+\infty}\ln(u)\frac{e^{-u/2}}{u}\mathrm{d}u}_{\text{int. by parts}}\\
&\quad+4\left(-4\ln(3)^2-4\ln(3)+8\ln(2)\ln(3)-8\mathrm{Li}_2\left(\frac{1}{3}\right)+\frac{2\pi^2}{3}\right)+4(\ln(2)-\gamma)^2\\
&\quad-4\left(-2\ln(2)^2+4(\gamma-1)\ln(2)-2\gamma+\frac{\pi^2}{3}\right)\\
&\quad+4(\ln(2)-\gamma)\left[4\mathrm{Ei}(-u)-2(u+2)e^{-u/2}\mathrm{Ei}\left(-\frac{u}{2}\right)-2e^{-u}\right]_0^{+\infty}\\
&\quad-8(\ln(2)-\gamma)-8(\ln(2)-\gamma)^2\\
&\quad+2(1-\gamma)-4\left[\mathrm{Ei}\left(-\frac{3u}{2}\right)-(u+1)e^{-u}\mathrm{Ei}\left(-\frac{u}{2}\right)-\frac{2}{3}e^{-3u/2}\right]_0^{+\infty}-2(\ln(2)-\gamma)\\
&\quad-(4+4(\ln(2)-\gamma))+2\left[4\mathrm{Ei}(-u)-2(u+2)e^{-u/2}\mathrm{Ei}\left(-\frac{u}{2}\right)-2e^{-u}\right]_0^{+\infty}+4(\ln(2)-\gamma) \qquad (101)
\end{aligned}
$$

$$
\begin{aligned}
&= 4\lim_{\epsilon\to 0}\ln(\epsilon)^2 + 8(\ln(2)-\gamma) + 8\left(\left[\ln(u)\mathrm{Ei}\left(-\frac{u}{2}\right)\right]_0^{+\infty} - \underbrace{\int_0^{+\infty}\frac{1}{u}\mathrm{Ei}\left(-\frac{u}{2}\right)\mathrm{d}u}_{(29)}\right)\\
&\quad - 16\ln(3)^2 - 16\ln(3) + 32\ln(2)\ln(3) - 32\mathrm{Li}_2\left(\frac{1}{3}\right) + \frac{8\pi^2}{3} + 4(\ln(2)-\gamma)^2\\
&\quad + 8\ln(2)^2 - 16(\gamma-1)\ln(2) + 8\gamma - \frac{4\pi^2}{3}\\
&\quad - 4\lim_{\epsilon\to 0}(\ln(2)-\gamma)\left(4\left(\ln(\epsilon)+\gamma\right) - 4\left(\ln(\epsilon)-\ln(2)+\gamma\right) - 2\right)\\
&\quad - 8(\ln(2)-\gamma) - 8(\ln(2)-\gamma)^2\\
&\quad + 2 - 2\gamma + 4\lim_{\epsilon\to 0}\left(\ln(\epsilon)+\ln(3)-\ln(2)+\gamma-(\ln(\epsilon)-\ln(2)+\gamma)-\frac{2}{3}\right) - 2(\ln(2)-\gamma)\\
&\quad - 4 - 4\left(\ln(2)-\gamma\right) - 2\lim_{\epsilon\to 0}\left(4\left(\ln(\epsilon)+\gamma\right) - 4\left(\ln(\epsilon)-\ln(2)+\gamma\right) - 2\right) + 4(\ln(2)-\gamma)
\end{aligned}
\tag{102}
$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\left(4\ln(\epsilon)^2 - 8\ln(\epsilon)^2 + 8(\ln(2)-\gamma)\ln(\epsilon) + 4\ln(\epsilon)^2 - 8(\ln(2)-\gamma)\ln(\epsilon) + 4(\ln(2)-\gamma)^2 + \frac{2\pi^2}{3}\right)\\
&\quad - 16\ln(3)^2 - 16\ln(3) + 32\ln(2)\ln(3) - 32\mathrm{Li}_2\left(\frac{1}{3}\right) + \frac{8\pi^2}{3}\\
&\quad + 8\ln(2)^2 - 16(\gamma-1)\ln(2) + 8\gamma - \frac{4\pi^2}{3}\\
&\quad - 4(\ln(2)-\gamma)\left(4\ln(2)-2\right) - 4(\ln(2)-\gamma)^2\\
&\quad + 2 - 2\gamma + 4\left(\ln(3)-\frac{2}{3}\right) - 2(\ln(2)-\gamma)\\
&\quad - 4 - 4\left(\ln(2)-\gamma\right) - 2\left(4\ln(2)-2\right) + 4(\ln(2)-\gamma)
\end{aligned}
\tag{103}
$$

$$
= 2\pi^2 - 32\mathrm{Li}_2\left(\frac{1}{3}\right) - 16\ln(3)^2 - 8\ln(2)^2 + 32\ln(2)\ln(3) - 12\ln(3) + 14\ln(2) - \frac{2}{3} \tag{104}
$$

Hence the result,

$$
\boxed{\int_{u=0}^{+\infty}\int_{v=0}^{+\infty}\int_{w=0}^{+\infty} uw\left|\ln\left(\frac{u}{v}\right)\right|\left|\ln\left(\frac{w}{u}\right)\right|p(u)p(v)p(w)\mathrm{d}u\mathrm{d}v\mathrm{d}w \approx 5.087625} \tag{105}
$$

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