Delay and Random Scattering Estimation with a Band-Limited Signal: Unconditional CRB and MLE

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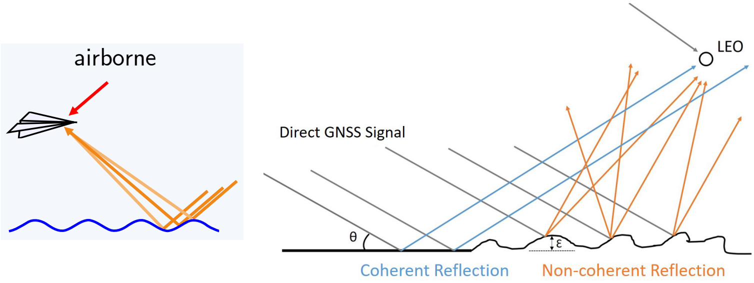
Intro

- Parameter estimation is a problem of interest in several remote sensing applications, e.g., radar, GNSS, or GNSS-R.
- Standard processing: conditional (deterministic) signal model → unknown parameters are assumed to be deterministic.
- More informative to consider a Gaussian random surface scattering.
- Gaussian scattering → unconditional (stochastic) signal model.I
- Goals of this contribution:
  - Closed-form unconditional CRB expressions for Gaussian source variance estimation, considering a generic band-limited signal.
  - Performance comparision with the corresponding unconditional MLEs.
  - Validation with 2 representative band-limited signals: GNSS and radar.

Motivation Example: GNSS-Based Reflectometry

GNSS as a bistatic radar of opportunity for Earth observation



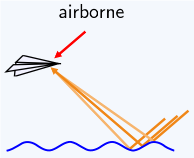
- Coherent: smooth surface (e.g., ice, calm water)
- Diffuse (noncoherent): roughness larger than the wavelength λ (e.g., ocean)

Source: Y. Wang and Y. J. Morton, "Coherent GNSS Reflection Signal Processing for High-Precision and High-Resolution Spaceborne Applications", IEEE Transactions on Geoscience and Remote Sensing, 2021

Conditional and Unconditional Signal Models

Consider the reception of a reflected signal (at the receiver down-looking antenna)

If the Doppler is correctly compensated:



Discrete baseband signal model, with K snapshots, each one with N samples at $T_s = 1/F_s$.

$$\mathbf{y}_k = \alpha_k \mathbf{s}(\tau) + \mathbf{n}_k, \ \mathbf{n}_k \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2 \mathbf{I}_N), \ k = 1, \ldots, K,$$

where $\mathbf{s}(\tau)$ are the delayed (τ) transmitted signal samples.

α_k is a complex amplitude that mainly depends on i) the reflecting surface, and ii) transmitter, receiver and propagation path.

Conditional and Unconditional Signal Models

Conditional signal model (CSM)	Unconditional signal model (USM)
α_k is a complex deterministic parameter $\mathbf{y}_k \sim \mathcal{CN}(\alpha_k \mathbf{s}(\tau), \sigma_n^2 \mathbf{I}_N), \;\; k = 1, \ldots, K,$ the unknown deterministic parameters are σ_n^2, $\boldsymbol{\alpha}^T = (\alpha_1, \ldots, \alpha_K)$, τ.	Consider a complex Gaussian prior for α_k $\alpha_k \sim \mathcal{CN}(0, \sigma_\alpha^2),$ $\mathbf{y}_k \sim \mathcal{CN}(\mathbf{0}, \sigma_\alpha^2 \mathbf{s}(\tau)\mathbf{s}^H(\tau) + \sigma_n^2 \mathbf{I}_N),$ the unknown parameters are $\boldsymbol{\theta}^T = (\sigma_n^2, \sigma_\alpha^2, \tau)$.

- ▷ CSM: well-known results for the conditional MLE (CMLE) and CRB (CRB_c)
- ▷ USM: we need the corresponding unconditional MLE (UMLE) and CRB (CRB_u)

CRB: Cramér-Rao bound (on the mean square error sense, i.e., minimum variance of any unbiased estimator)
MLE: maximum likelihood estimator

Conditional and Unconditional MLEs

Noncoherent conditional MLEs

$$\widehat{\tau}_c = \arg\max_{\tau} \sum_{k=1}^{K} |r_k(\tau)|^2,$$

$$\widehat{\sigma}^2_{n,c} = \frac{1}{KN} \sum_{k=1}^{K} \left(\|\mathbf{y}_k\|^2 - \frac{|r_k(\widehat{\tau}_c)|^2}{\|\mathbf{s}\|^2} \right),$$

$$\widehat{\alpha}_{k,c} = \frac{r_k(\widehat{\tau}_c)}{\|\mathbf{s}\|^2} \quad \text{(for each snapshot).}$$

$\|\mathbf{s}\|^2 \triangleq \mathbf{s}^H\mathbf{s}$, and cross-correlation function $r_k(\tau) = \mathbf{s}^H(\tau)\mathbf{y}_k$.

With these estimates we can build the scattering (sample) variance estimates as (MLE invariance principle).

$$\widehat{\sigma}^2_{\alpha,c} = \frac{1}{K-1} \sum_{k=1}^{K} \left(\widehat{\alpha}_{k,c} - \frac{1}{K} \sum_{m=1}^{K} \widehat{\alpha}_{m,c} \right)^2.$$

Conditional and Unconditional MLEs

Unconditional MLE

$$\widehat{\tau}_u = \arg\min_{\tau}\{C_u(\tau)\},$$

$$C_u(\tau) = \left(\sum_{k=1}^{K}\left(\|\mathbf{y}_k\|^2 - \frac{|r_k(\tau)|^2}{\|\mathbf{s}\|^2}\right)\right)^{N-1}\left(\sum_{k=1}^{K}\left|r_k(\tau)\right|^2\right),$$

$$\widehat{\sigma}^2_{n,u} = \frac{1}{K(N-1)}\sum_{k=1}^{K}\left(\|\mathbf{y}_k\|^2 - \frac{|r_k(\widehat{\tau}_u)|^2}{\|\mathbf{s}\|^2}\right),$$

and the Gaussian scattering variance UMLE is,

$$\widehat{\sigma}^2_{\alpha,u} = \frac{1}{K(N-1)\|\mathbf{s}\|^2}\sum_{k=1}^{K}\left(N\frac{|r_k(\widehat{\tau}_u)|^2}{\|\mathbf{s}\|^2} - \|\mathbf{y}_k\|^2\right).$$

Unconditional Cramér-Rao Bounds

$$\mathrm{CRB}_u(\tau) = \frac{1+\beta}{2K\beta^2\left(\frac{W_{33}}{w_1} - \frac{|w_3|^2}{w_1^2}\right)} \; ; \; \mathrm{CRB}_u(\sigma_\alpha^2) = \frac{\left(\sigma_\alpha^2\right)^2\left(1+(N-1)(1+\beta)^2\right)}{K(N-1)\beta^2},$$

with $\beta = \sigma_\alpha^2\|\mathbf{s}\|^2/\sigma_n^2$, $w_1 = \|\mathbf{s}\|^2/F_s$, $w_3 = \mathbf{s}^H\mathbf{\Lambda}\mathbf{s}/F_s$, $W_{33} = F_s\mathbf{s}^H\mathbf{V}\mathbf{s}$,

$$[\mathbf{\Lambda}]_{k,\ell} = \left\{0 \text{ if } k=\ell, \frac{(-1)^{|k-\ell|}}{k-\ell} \text{ else}\right\} \; ; \; [\mathbf{V}]_{k,\ell} = \left\{\pi^2/3 \text{ if } k=\ell, \frac{2(-1)^{|k-\ell|}}{(k-\ell)^2} \text{ else.}\right\}.$$

Key advantage of this formulation: only depends on the baseband signal samples

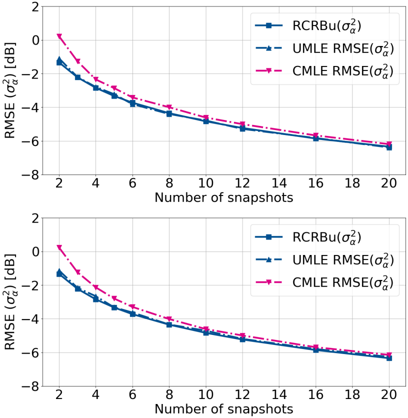
CRB for the CSM

$$\mathrm{CRB}_c(\tau) = \frac{1}{2K\gamma}\frac{1}{\frac{W_{33}}{w_1} - \frac{|w_3|^2}{w_1^2}},$$

where $\gamma = \frac{1}{K}\sum_{k=1}^{K}|\alpha_k|^2\|\mathbf{s}\|^2/\sigma_n^2$.
For a large number of snapshots K, γ tends to β, then $\mathrm{CRB}_u(\tau) \geq \mathrm{CRB}_c(\tau)$, with equality in the large number of snapshots and/or high SNR regimes.

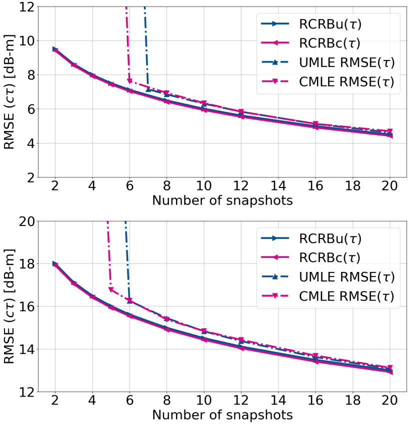
Validation

Two representative band-limited signals are considered: 1) a GPS L1 C/A signal (sampled at 4 MHz), and 2) a radar chirp signal (bandwidth 250 kHz).



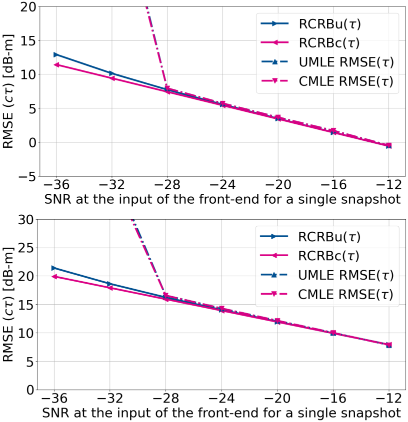
RMSE of the variance estimate $\widehat{\sigma}_\alpha^2$ for both UMLE and CMLE, together with the root CRB_u. (Top) GPS signal; (Bottom) Chirp signal. With SNR=-22 at the input of the receiver.

Validation



RMSE of the delay estimate $\hat{\tau}$ for both UMLE and CMLE, together with the root CRB_u and CRB_c. (Top) GPS signal; (Bottom) Chirp signal. With SNR=-22 at the input of the receiver.

Validation



RMSE of the delay estimate $\hat{\tau}$ for both UMLE and CMLE, together with the root CRB_u and CRB_c as a function of the SNR, for $K = 20$ snapshots. (Top) GPS signal; (Bottom) Chirp signal.

Conclusions and Future Research

Conclusions

- ▷ Derivation of the unconditional CRB for delay and Gaussian scattering variance.
- ▷ General closed-form CRB expressions for any band-limited signal, which only depend on the signal samples.
- ▷ Implementation of the corresponding ML estimators.
- ▷ Validation with 2 representative signals: GNSS and radar.

Future research

- ▷ Delay/Doppler estimation.
- ▷ Mixed conditional/unconditional models.
- ▷ Reflecting surface characterization.
- ▷ Multi-receiver case.

Thanks for your attention