Technical Note
Developments for MCRB Computation in Multipath Scenarios

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May 3, 2023

1 Derivation of Convolution Terms Using the Fourier Transform Properties

Let $s(t)$ and $r(t)$ be band-limited signals of bandwidth smaller than B sampled at frequency F_s over $N = N_2 - N_1$ samples. The aim of this note is to provide details on the evaluation of the following integral terms:

$$w_1 = \int_{\mathbb{R}} s(t-\tau) r(t-\tau_k)^* e^{-j2\pi f_c (b(t-\tau) - b_k(t-\tau_k))} \mathrm{d}t, \tag{1}$$

$$w_2 = \int_{\mathbb{R}} (t-\tau) s(t-\tau) r(t-\tau_k)^* e^{-j2\pi f_c (b(t-\tau) - b_k(t-\tau_k))} \mathrm{d}t, \tag{2}$$

$$w_3 = \int_{\mathbb{R}} (t-\tau)^2 s(t-\tau) r(t-\tau_k)^* e^{-j2\pi f_c (b(t-\tau) - b_k(t-\tau_k))} \mathrm{d}t, \tag{3}$$

$$w_4 = \int_{\mathbb{R}} s^{(1)}(t-\tau) r(t-\tau_k)^* e^{-j2\pi f_c (b(t-\tau) - b_k(t-\tau_k))} \mathrm{d}t, \tag{4}$$

$$w_5 = \int_{\mathbb{R}} (t-\tau) s^{(1)}(t-\tau) r(t-\tau_k)^* e^{-j2\pi f_c (b(t-\tau) - b_k(t-\tau_k))} \mathrm{d}t, \tag{5}$$

$$w_6 = \int_{\mathbb{R}} s^{(2)}(t-\tau) r(t-\tau_k)^* e^{-j2\pi f_c (b(t-\tau) - b_k(t-\tau_k))} \mathrm{d}t, \tag{6}$$

where τ is a time delay, $(1-b)$ is a dilatation term due to Doppler effect, f_c is carrier frequency and the superscript $^{(i)}$ refers to the i-th time derivative of signal $s(t)$.

1.1 Prior Considerations

First the Fourier transform of a set of functions are to be evaluated. Remembering that the signal is band-limited of band $B \leq F_s$, one has:

$$s(t) \rightleftharpoons \mathrm{FT}\left\{s(t)\right\}(f) \triangleq S(f) = \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n T_s} \right) 1_{\left[-\frac{F_s}{2};\frac{F_s}{2}\right]} \tag{7}$$

In order to tackle the issue that may come from the spectral shift due to Doppler effect. One simply needs to take F_s large enough so that $\frac{F_s}{2} \geq \frac{B}{2} + f_c \max\{|b|, |b_k|, |b - b_k|\}$.

A first expression is a simple application of the frequency shift relation when using the Fourier transform of a signal multiplied by a complex time-varying exponential.

$$s(t)e^{j2\pi f_c bt} \rightleftharpoons \text{FT}\left\{s(t)e^{j2\pi f_c bt}\right\}(f) \triangleq S(f - f_c b) \tag{8}$$

Then, let s_1 be defined by $s_1(t;b) = s(t)e^{j2\pi f_c bt}$, it is known that

$$ts_1(t;b) \rightleftharpoons \frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\underbrace{\left(\text{FT}\left\{s_1(t;b)\right\}(f)\right)}_{(8)} \tag{9}$$

therefore

$$ts(t)e^{j2\pi f_c bt} \rightleftharpoons \frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f - f_c b)\right) \tag{10}$$

Similarly

$$t^2 s(t)e^{j2\pi f_c bt} \rightleftharpoons \left(\frac{j}{2\pi}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}f^2}\left(S(f - f_c b)\right) \tag{11}$$

Besides, with the superscript $^{(1)}$ referring to the first time derivative,

$$s_1^{(1)}(t;b) \triangleq \frac{\mathrm{d}}{\mathrm{d}t}(s_1(t;b)) = s^{(1)}(t)e^{j2\pi f_c bt} + (j2\pi f_c b)s_1(t;b)$$
$$\Leftrightarrow s^{(1)}(t)e^{j2\pi f_c bt} = s_1^{(1)}(t;b) - (j2\pi f_c b)s_1(t;b)$$

Then, knowing the Fourier transform of the k-th time derivative of a function

$$\text{FT}\left\{s^{(k)}(t)\right\}(f) \triangleq (j2\pi f)^k S(f) \tag{12}$$

one directly gets

$$s^{(1)}(t)e^{j2\pi f_c bt} \rightleftharpoons j2\pi\left(f - f_c b\right)S(f - f_c b) \tag{13}$$

Now, if s_2 is defined as $s_2(t;b) = ts(t)e^{j2\pi f_c bt}$,

$$s_2^{(1)}(t;b) = s_1(t;b) + ts^{(1)}(t)e^{j2\pi f_c bt} + (j2\pi f_c b)s_2(t;b)$$
$$\Leftrightarrow ts^{(1)}(t)e^{j2\pi f_c bt} = -\underbrace{s_1(t;b)}_{(8)} + \underbrace{s_2^{(1)}(t;b)}_{(12)} - (j2\pi f_c b)\underbrace{s_2(t;b)}_{(10)}$$

therefore,

$$ts^{(1)}(t)e^{j2\pi f_c bt} \rightleftharpoons -S(f - f_c b) - (f - f_c b)\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f - f_c b)\right) \tag{14}$$

Finally, by taking again s_1 as $s_1(t;b) = s(t)e^{j2\pi f_c bt}$,

$$s_1^{(2)}(t;b) = s^{(2)}(t)e^{j2\pi f_c bt} + 2(j2\pi f_c b)s^{(1)}(t)e^{j2\pi f_c bt} + (j2\pi f_c b)^2 s_1(t;b) \tag{15}$$

$$\Leftrightarrow s^{(2)}(t)e^{j2\pi f_c bt} = \underbrace{s_1^{(2)}(t;b)}_{(12)} - (j4\pi f_c b)\underbrace{s^{(1)}(t)e^{j2\pi f_c bt}}_{(13)} + 4\pi^2(f_c b)^2\underbrace{s_1(t;b)}_{(8)} \tag{16}$$

one obtains,

$$s^{(2)}(t)e^{j2\pi f_c bt} \rightleftharpoons (j2\pi f)^2 S(f - f_c b) + 8\pi^2 f_c b(f - f_c b)S(f - f_c)) + 4\pi^2(f_c b)^2 S(f - f_c b) \tag{17}$$

$$\rightleftharpoons -4\pi^2\left(f - f_c b\right)^2 S(f - f_c b) \tag{18}$$

1.2 Evaluation of the Integrals

1.2.1 Integral w_1

$$\begin{aligned} w_1 &= \int_{\mathbb{R}} s(t-\tau) r(t-\tau_k)^* e^{-j\omega_c(b(t-\tau)-b_k(t-\tau_k))} \mathrm{d}t \\ &= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} s(u) r(u-\Delta\tau_k)^* e^{j\omega_c \Delta b_k u} \mathrm{d}u \end{aligned}$$

with $\Delta\tau_k \triangleq \tau_k - \tau$ and $\Delta b_k \triangleq b_k - b$. Then, using the Fourier transform properties over the hermitian product,

$$\begin{aligned} w_1 e^{j\omega_c b_k \Delta\tau_k} &= \int_{\mathbb{R}} \underbrace{s(u) e^{j\omega_c \Delta b_k u}}_{(8)} \left(r(u-\Delta\tau_k)\right)^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} S(f - f_c\Delta b_k) \left(R(f) e^{-j2\pi f \Delta\tau_k}\right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi(f-f_c\Delta b_k)nT_s}\right) e^{j2\pi f \Delta\tau_k} \left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s}\right)^* \mathrm{d}f \\
&= \frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n}\right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left(\sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n}\right)^* \mathrm{d}f \\
&= \frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \boldsymbol{\nu}(f)^*\right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left(\mathbf{r}^H \boldsymbol{\nu}(f)\right) \mathrm{d}f \\
&= \frac{1}{F_s}\mathbf{r}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \mathrm{d}f\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \\
&= \frac{1}{F_s}\mathbf{r}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \end{aligned}$$

Hence

$$\boxed{w_1 = \frac{1}{F_s}\mathbf{r}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s}\, e^{-j\omega_c b_k \Delta\tau_k}} \tag{19}$$

with

$$\boldsymbol{\nu}(f) = \begin{pmatrix} \dots & e^{j2\pi f n} & \dots \end{pmatrix}^T_{N_1 \le n \le N_2} \tag{20}$$

$$\mathbf{U}(p) = \mathrm{diag}\begin{pmatrix} \dots & e^{-j2\pi p n} & \dots \end{pmatrix}_{N_1 \le n \le N_2} \tag{21}$$

$$\mathbf{V}^{\Delta,0}(q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f) e^{-j2\pi f q} \mathrm{d}f \tag{22}$$

and

$$\left[\mathbf{V}^{\Delta,0}\left(q\right)\right]_{k,l} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi f(k-l-q)} \mathrm{d}f$$
$$= \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$
$$= \frac{\sin\left(\pi(k-l-q)\right)}{\pi(k-l-q)} = \mathrm{sinc}\left(k-l-q\right) \tag{23}$$

1.2.2 Integral w_2

$$w_2 = \int_{\mathbb{R}} (t-\tau)s(t-\tau)r(t-\tau_k)^* e^{-j\omega_c(b(t-\tau)-b_k(t-\tau_k))}\mathrm{d}t$$
$$= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} us(u)r(u-\Delta\tau_k)^* e^{j\omega_c \Delta b_k u}\mathrm{d}u$$

Therefore,

$$w_2 e^{j\omega_c b_k \Delta\tau_k} = \int_{\mathbb{R}} \underbrace{us(u)e^{j\omega_c \Delta b_k u}}_{(10)} \left(r(u-\Delta\tau_k)\right)^* \mathrm{d}u$$
$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f-f_c\Delta b_k)\right)\right)\left(R(f)e^{-j2\pi f\Delta\tau_k}\right)^* \mathrm{d}f$$
$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{1}{F_s}\frac{j}{2\pi}(-j2\pi T_s)\sum_{n=N_1}^{N_2} s(nT_s)ne^{-j2\pi(f-f_c\Delta b_k)nT_s}\right) e^{j2\pi f\Delta\tau_k}$$
$$\times \left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} r(nT_s)e^{-j2\pi f nT_s}\right)^* \mathrm{d}f$$
$$= \frac{1}{F_s^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=N_1}^{N_2} s(nT_s)ne^{-j2\pi fn}e^{j2\pi\frac{f_c\Delta b_k}{F_s}n}\right) e^{j2\pi f\frac{\Delta\tau_k}{T_s}} \left(\sum_{n=N_1}^{N_2} r(nT_s)e^{-j2\pi fn}\right)^* \mathrm{d}f$$
$$= \frac{1}{F_s^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T\mathbf{D}\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\boldsymbol{\nu}(f)^*\right) e^{j2\pi f\frac{\Delta\tau_k}{T_s}} \left(\mathbf{r}^H\boldsymbol{\nu}(f)\right)\mathrm{d}f$$
$$= \frac{1}{F_s^2}\mathbf{r}^H\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f)e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\mathrm{d}f\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{D}\mathbf{s}$$
$$= \frac{1}{F_s^2}\mathbf{r}^H\mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{D}\mathbf{s}$$

Hence

$$\boxed{w_2 = \frac{1}{F_s^2}\mathbf{r}^H\mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{D}\mathbf{s}\, e^{-j\omega_c b_k\Delta\tau_k}} \tag{24}$$

with $\mathbf{U}$ and $\mathbf{V}^{\Delta,0}$, defined in (21) and (22), respectively, and with

$$\mathbf{D} = \mathrm{diag}\begin{pmatrix} \dots & n & \dots \end{pmatrix}_{N_1\leq n\leq N_2} \tag{25}$$

1.2.3 Integral w_3

$$
\begin{aligned}
w_3 &= \int_{\mathbb{R}} (t-\tau)^2 s(t-\tau) r(t-\tau_k)^* e^{-j\omega_c(b(t-\tau)-b_k(t-\tau_k))} \mathrm{d}t \\
&= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} u^2 s(u) r(u-\Delta\tau_k)^* e^{j\omega_c \Delta b_k u} \mathrm{d}u
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
w_3 e^{j\omega_c b_k \Delta\tau_k} &= \int_{\mathbb{R}} \underbrace{u^2 s(u) e^{j\omega_c \Delta b_k u}}_{(11)} \left(r(u-\Delta\tau_k)\right)^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\left(\frac{j}{2\pi}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}f^2} \left(S(f - f_c \Delta b_k)\right) \right) \left(R(f) e^{-j2\pi f \Delta\tau_k}\right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{1}{F_s} \left(\frac{j}{2\pi}\right)^2 (-j2\pi T_s)^2 \sum_{n=N_1}^{N_2} s(nT_s) n^2 e^{-j2\pi(f - f_c\Delta b_k) n T_s} \right) e^{j2\pi f \Delta\tau_k} \\
&\quad \times \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s^3} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=N_1}^{N_2} s(nT_s) n^2 e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n} \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left(\sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s^3} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{D}^2 \mathbf{U}\left(-\frac{f_c \Delta b_k}{F_s}\right) \boldsymbol{\nu}(f)^* \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left(\mathbf{r}^H \boldsymbol{\nu}(f)\right) \mathrm{d}f \\
&= \frac{1}{F_s^3} \mathbf{r}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(-\frac{f_c \Delta b_k}{F_s}\right) \mathbf{D}^2 \mathbf{s} \\
&= \frac{1}{F_s^3} \mathbf{r}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c \Delta b_k}{F_s}\right) \mathbf{D}^2 \mathbf{s}
\end{aligned}
$$

Hence

$$
\boxed{w_3 = \frac{1}{F_s^3} \mathbf{r}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c \Delta b_k}{F_s}\right) \mathbf{D}^2 \mathbf{s}\, e^{-j\omega_c b_k \Delta\tau_k}} \tag{26}
$$

with $\mathbf{U}$, $\mathbf{V}^{\Delta,0}$ and $\mathbf{D}$ defined in (21), (22) and (25) respectively.

1.2.4 Integral w_4

$$
\begin{aligned}
w_4 &= \int_{\mathbb{R}} s^{(1)}(t-\tau) r(t-\tau_k)^* e^{-j\omega_c(b(t-\tau)-b_k(t-\tau_k))} \mathrm{d}t \\
&= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} s^{(1)}(u) r(u-\Delta\tau_k)^* e^{j\omega_c \Delta b_k u} \mathrm{d}u
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
w_4 e^{j\omega_c b_k \Delta\tau_k} &= \int_{\mathbb{R}} \underbrace{s^{(1)}(u) e^{j\omega_c \Delta b_k u}}_{(13)} \left(r(u-\Delta\tau_k)\right)^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(j2\pi(f - f_c\Delta b_k) S(f - f_c\Delta b_k)\right) \left(R(f) e^{-j2\pi f \Delta\tau_k}\right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(j2\pi(f - f_c\Delta b_k) \frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi(f - f_c\Delta b_k) n T_s} \right) e^{j2\pi f \Delta\tau_k} \\
&\quad \times \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(j2\pi(fF_s - f_c\Delta b_k) \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n} \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \\
&\quad \times \left(\sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(j2\pi(fF_s - f_c\Delta b_k) \mathbf{s}^T \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \boldsymbol{\nu}(f)^* \right) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left(\mathbf{r}^H \boldsymbol{\nu}(f)\right) \mathrm{d}f \\
&= \mathbf{r}^H \left(j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \\
&\quad - \frac{j2\pi f_c \Delta b_k}{F_s} \mathbf{r}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \\
&= \mathbf{r}^H \mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} - \frac{j2\pi f_c \Delta b_k}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s}
\end{aligned}
$$

Hence

$$
\boxed{
\begin{aligned}
w_4 = &\left(\mathbf{r}^H \mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} - \frac{j\omega_c \Delta b_k}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right) \right. \\
&\quad \left. \times \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \right) e^{-j\omega_c b_k \Delta\tau_k}
\end{aligned}
} \tag{27}
$$

with $\mathbf{U}$ and $\mathbf{V}^{\Delta,0}$ defined in (21) and (22) and

$$
\mathbf{V}^{\Delta,1}(q) = j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f q} \mathrm{d}f \tag{28}
$$

and

$$
\begin{aligned}
\left[\mathbf{V}^{\Delta,1}(q)\right]_{k,l} &= j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f e^{j2\pi f(k-l-q)} \mathrm{d}f \\
&= j2\pi \left(\left[\frac{f e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \mathrm{d}f \right) \\
&= \frac{j2\pi}{j2\pi(k-l-q)} \left(\left[\frac{1}{2} e^{j\pi(k-l-q)} - \left(-\frac{1}{2}\right) e^{-j\pi(k-l-q)} \right] - \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \right) \\
&= \frac{1}{k-l-q} \left(\cos\left(\pi(k-l-q)\right) - \text{sinc}\left(k-l-q\right) \right) \qquad (29)
\end{aligned}
$$

1.2.5 Integral w_5

$$
\begin{aligned}
w_5 &= \int_{\mathbb{R}} (t-\tau) s^{(1)}(t-\tau) r(t-\tau_k)^* e^{-j\omega_c(b(t-\tau)-b_k(t-\tau_k))} \mathrm{d}t \\
&= e^{-j\omega_c b_k \Delta\tau_k} \int_{\mathbb{R}} u s^{(1)}(u) r(u-\Delta\tau_k)^* e^{j\omega_c \Delta b_k u} \mathrm{d}u
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
w_5 e^{j\omega_c b_k \Delta\tau_k} &= \int_{\mathbb{R}} \underbrace{u s^{(1)}(u) e^{j\omega_c \Delta b_k u}}_{(14)} \left(r(u-\Delta\tau_k)\right)^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(-S(f-f_c\Delta b_k) - (f-f_c\Delta b_k)\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f-f_c\Delta b_k)\right)\right)\left(R(f)e^{-j2\pi f\Delta\tau_k}\right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \Bigg(-\left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} s(nT_s)e^{-j2\pi(f-f_c\Delta b_k)nT_s}\right) \\
&\qquad - (f-f_c\Delta b_k)\left(\frac{1}{F_s}(-j2\pi T_s)\sum_{n=N_1}^{N_2} s(nT_s)ne^{-j2\pi(f-f_c\Delta b_k)nT_s}\right)\Bigg) \\
&\qquad \times e^{j2\pi f\Delta\tau_k}\left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} r(nT_s)e^{-j2\pi f nT_s}\right)^* \mathrm{d}f \\
&= -\frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=N_1}^{N_2} s(nT_s)e^{-j2\pi f n}e^{j2\pi\frac{f_c\Delta b_k}{F_s}n}\right) e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\left(\sum_{n=N_1}^{N_2} r(nT_s)e^{-j2\pi f n}\right)^* \mathrm{d}f \\
&\qquad + \frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} j2\pi f\left(\sum_{n=N_1}^{N_2} s(nT_s)ne^{-j2\pi f n}e^{j2\pi\frac{f_c\Delta b_k}{F_s}n}\right) e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\left(\sum_{n=N_1}^{N_2} r(nT_s)e^{-j2\pi f n}\right)^* \mathrm{d}f \\
&\qquad - j2\pi\frac{f_c\Delta b_k}{F_s^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=N_1}^{N_2} s(nT_s)ne^{-j2\pi f n}e^{j2\pi\frac{f_c\Delta b_k}{F_s}n}\right) e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\left(\sum_{n=N_1}^{N_2} r(nT_s)e^{-j2\pi f n}\right)^* \mathrm{d}f \\
&= -\frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\boldsymbol{\nu}(f)^*\right) e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\left(\mathbf{r}^H\boldsymbol{\nu}(f)\right)\mathrm{d}f \\
&\qquad + \frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} j2\pi f\left(\mathbf{s}^T\mathbf{D}\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\boldsymbol{\nu}(f)^*\right) e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\left(\mathbf{r}^H\boldsymbol{\nu}(f)\right)\mathrm{d}f \\
&\qquad - j2\pi\frac{f_c\Delta b_k}{F_s^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T\mathbf{D}\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\boldsymbol{\nu}(f)^*\right) e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\left(\mathbf{r}^H\boldsymbol{\nu}(f)\right)\mathrm{d}f \\
&= -\frac{1}{F_s}\mathbf{r}^H\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f)\boldsymbol{\nu}(f)^H e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\mathrm{d}f\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{s} \\
&\qquad + \frac{1}{F_s}\mathbf{r}^H\left(j2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} f\boldsymbol{\nu}(f)\boldsymbol{\nu}(f)^H e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\mathrm{d}f\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{D}\mathbf{s} \\
&\qquad - j2\pi\frac{f_c\Delta b_k}{F_s^2}\mathbf{r}^H\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f)\boldsymbol{\nu}(f)^H e^{j2\pi f\frac{\Delta\tau_k}{T_s}}\mathrm{d}f\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{D}\mathbf{s} \\
&= -\frac{1}{F_s}\mathbf{r}^H\mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{s} + \frac{1}{F_s}\mathbf{r}^H\mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau_k}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{D}\mathbf{s} \\
&\qquad - j2\pi\frac{f_c\Delta b_k}{F_s^2}\mathbf{r}^H\mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{D}\mathbf{s}
\end{aligned}
$$

Hence

$$
\boxed{\begin{aligned} w_5 = &\left(-\frac{1}{F_s}\mathbf{r}^H\mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{s} + \frac{1}{F_s}\mathbf{r}^H\mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau_k}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{Ds}\right. \\ &\left. -j\frac{\omega_c\Delta b_k}{F_s^2}\mathbf{r}^H\mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right)\mathbf{Ds}\right) e^{-j\omega_c b_k\Delta\tau_k} \end{aligned}} \tag{30}
$$

with $\mathbf{U}$, $\mathbf{V}^{\Delta,0}$, $\mathbf{V}^{\Delta,1}$ and $\mathbf{D}$ defined in (21), (22), (28) and (25).

1.2.6 Integral w_6

$$
\begin{aligned} w_6 &= \int_{\mathbb{R}} s^{(2)}(t-\tau)r(t-\tau_k)^* e^{-j\omega_c(b(t-\tau)-b_k(t-\tau_k))}\mathrm{d}t \\ &= e^{-j\omega_c b_k\Delta\tau_k}\int_{\mathbb{R}} s^{(2)}(u)r(u-\Delta\tau_k)^* e^{j\omega_c\Delta b_k u}\mathrm{d}u \end{aligned}
$$

$$
\begin{aligned}
w_6 e^{j\omega_c b_k \Delta\tau_k} &= \int_{\mathbb{R}} \underbrace{s^{(2)}(u) e^{j\omega_c \Delta b_k u}}_{(18)} \left(r(u-\Delta\tau_k)\right)^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(-4\pi^2 (f - f_c\Delta b_k)^2 S(f - f_c \Delta b_k)\right) \left(R(f) e^{-j2\pi f \Delta\tau_k}\right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\left(-4\pi^2 f^2 + 8\pi^2 f f_c \Delta b_k - 4\pi^2 (f_c \Delta b_k)^2\right) \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi(f - f_c\Delta b_k) n T_s} \right) \right) \\
&\quad \times e^{j2\pi f \Delta\tau_k} \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n T_s} \right)^* \mathrm{d}f \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(-4\pi^2 (fF_s)^2 + 8\pi^2 (fF_s) f_c \Delta b_k - 4\pi^2 (f_c \Delta b_k)^2\right) \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{j2\pi \frac{f_c \Delta b_k}{F_s} n} \right) \right) \\
&\quad \times e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} r(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f F_s \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(-F_s (4\pi^2 f^2) - j4\pi f_c \Delta b_k (j2\pi f) - 4\pi^2 \frac{(f_c\Delta b_k)^2}{F_s} \right) \left(\mathbf{s}^T \mathbf{U}\left(-\frac{f_c \Delta b_k}{F_s}\right) \boldsymbol{\nu}(f)^* \right) \right) \\
&\quad \times e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \left(\mathbf{r}^H \boldsymbol{\nu}(f) \right) \mathrm{d}f \\
&= -F_s \mathbf{r}^H \left(4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \\
&\quad - j4\pi f_c \Delta b_k \mathbf{r}^H \left(j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \\
&\quad - 4\pi^2 \frac{(f_c\Delta b_k)^2}{F_s} \mathbf{r}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f) \boldsymbol{\nu}(f)^H e^{j2\pi f \frac{\Delta\tau_k}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \\
&= -F_s \mathbf{r}^H \mathbf{V}^{\Delta,2}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} - j4\pi f_c \Delta b_k \mathbf{r}^H \mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \\
&\quad - 4\pi^2 \frac{(f_c\Delta b_k)^2}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s}
\end{aligned}
$$

Hence

$$
\boxed{
\begin{aligned}
w_6 = \Bigg(& -F_s \mathbf{r}^H \mathbf{V}^{\Delta,2}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} - j2\omega_c \Delta b_k \mathbf{r}^H \mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \\
& - \frac{(\omega_c \Delta b_k)^2}{F_s} \mathbf{r}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau_k}{T_s}\right) \mathbf{U}\left(-\frac{f_c\Delta b_k}{F_s}\right) \mathbf{s} \Bigg) e^{-j\omega_c b_k \Delta\tau_k}
\end{aligned}
}
\tag{31}
$$

with $\mathbf{U}$, $\mathbf{V}^{\Delta,0}$ and $\mathbf{V}^{\Delta,1}$ defined in (21), (22) and (28) and

$$
\mathbf{V}^{\Delta,2}(q) = 4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 \boldsymbol{\nu}(f) \boldsymbol{\nu}^H(f) e^{-j2\pi f q} \mathrm{d}f \tag{32}
$$

and

$$
\begin{aligned}
\left[\mathbf{V}^{\Delta,2}(q)\right]_{k,l} &= 4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 e^{j2\pi f(k-l-q)} \mathrm{d}f \\
&= 4\pi^2 \left(\left[\frac{f^2 e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{2f e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \mathrm{d}f \right) \\
&= \frac{4\pi^2}{j2\pi(k-l-q)} \frac{1}{4} \left[e^{j\pi(k-l-q)} - e^{-j\pi(k-l-q)} \right] \\
&\quad - \frac{8\pi^2}{j2\pi(k-l-q)} \left(\left[\frac{f e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \mathrm{d}f \right) \\
&= \pi^2 \mathrm{sinc}\,(k-l-q) - \frac{8\pi^2}{(j2\pi(k-l-q))^2} \\
&\quad \times \left(\left[\frac{1}{2} e^{j\pi(k-l-q)} - \left(-\frac{1}{2}\right) e^{-j\pi(k-l-q)} \right] - \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \right) \\
&= \pi^2 \mathrm{sinc}\,(k-l-q) + \frac{8\pi^2}{4\pi^2(k-l-q)^2} \\
&\quad \times (\cos(\pi(k-l-q)) - \mathrm{sinc}\,(k-l-q)) \\
&= \pi^2 \mathrm{sinc}\,(k-l-q) + 2\frac{cos\,(\pi(k-l-q)) - \mathrm{sinc}\,(k-l-q)}{(k-l-q)^2}
\end{aligned} \tag{33}
$$

1.3 Matrix Properties

Based on the definitions of matrices $\mathbf{V}^{\Delta,0}$, $\mathbf{V}^{\Delta,1}$, $\mathbf{V}^{\Delta,2}$ and $\mathbf{U}$, one can do the following remarks:

- $\left(\mathbf{V}^{\Delta,0}(q)\right)^H = \mathbf{V}^{\Delta,0}(-q)$,
- $\left(\mathbf{V}^{\Delta,1}(q)\right)^H = -\mathbf{V}^{\Delta,1}(-q)$,
- $\left(\mathbf{V}^{\Delta,2}(q)\right)^H = \mathbf{V}^{\Delta,2}(-q)$,
- $(\mathbf{U}(p))^H = \mathbf{U}(-p)$.

2 Kullback-Leibler Divergence and Mispecified Maximum Likelihood Estimator

2.1 Signal Models

In this section two signal models are considered. One, referred to as the true model considers the reception of two sources embedded in an additive white Gaussian noise:

$$
\mathbf{x} = \alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1 + \mathbf{w},\ \mathbf{w} \sim \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_N)\ , \tag{34}
$$

with, for $i \in \{0,1\}$, $\alpha_i = \rho_i e^{j\phi_i}$ and, for $n \in [N_1, N_2]$,

$$\begin{aligned}
\mathbf{x}^T &= (\ldots,\ x(nT_s),\ \ldots)\ , \\
\mathbf{a}_i^T &= \mathbf{a}(\boldsymbol{\eta}_i)^T = (\ldots,\ a(nT_s;\boldsymbol{\eta}_i),\ \ldots)\ , \\
\mathbf{w}^T &= (\ldots,\ w(nT_s),\ \ldots)\ .
\end{aligned}$$

$\boldsymbol{\eta}_i$ is a vector of unknown parameters (delay and Doppler parameters for instance) that parameterized the signal of interest $\mathbf{a}$ and $\boldsymbol{\theta}_i^T = [\boldsymbol{\eta}_i^T, \rho_i, \phi_i]$ is the concatenated vector of parameters the i-th contributor. Consequently, the true data model PDF, noted $p_{\mathbf{x}}(\mathbf{x}|\boldsymbol{\theta}_0, \boldsymbol{\theta}_1)$, is written as:

$$p_{\mathbf{x}}(\mathbf{x}|\boldsymbol{\theta}_0, \boldsymbol{\theta}_1) = \mathcal{CN}(\alpha_0 \mathbf{a}_0 + \alpha_1 \mathbf{a}_1, \sigma_n^2 \mathbf{I}_N). \tag{35}$$

The second signal model, referred to as psuedotrue model considers the reception of a single source with an additivie white Gaussian noise:

$$\mathbf{x} = \alpha_{pt}\mathbf{a}_{pt} + \mathbf{w},\ \ \mathbf{w} \sim \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_N)\ , \tag{36}$$

with, $\alpha_{pt} = \rho_{pt} e^{j\phi_{pt}}$ and, for $n \in [N_1, N_2]$,

$$\mathbf{a}_{pt}^T = \mathbf{a}(\boldsymbol{\eta}_{pt})^T = \left(\ldots,\ a(nT_s;\boldsymbol{\eta}_{pt}),\ \ldots\right)\ ,$$

and the subscript pt refers to *psuedotrue* which will necessarily depend on the true values $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$. Then, the misspecified data model PDF, noted $f_{\mathbf{x}}(\mathbf{x}|\boldsymbol{\theta}_{pt})$ is written as:

$$f_{\mathbf{x}}(\mathbf{x}|\boldsymbol{\theta}_{pt}) = \mathcal{CN}(\alpha_{pt}\mathbf{a}_{pt}, \sigma_n^2 \mathbf{I}_N) \tag{37}$$

where $\boldsymbol{\theta}_{pt} = [\boldsymbol{\eta}_{pt}, \rho_{pt}, \phi_{pt}]^T$ is the vector of pseudotrue parameters.

2.2 Kullback-Leibler Divergence Computation

In the case of a misspecified scenario, one uses a the single source model (37) whereas the true model is a dual source model (35). In this case, the pseudotrue parameters $\boldsymbol{\theta}_{pt}$ are the values of the parameters $\boldsymbol{\theta} = [\boldsymbol{\eta}^T, \rho, \phi]^T$ that minimise the Kullback-Leibler Divergence (KLD) between the true and misspecified distribution models.

$$D(p_{\mathbf{x}}||f_{\mathbf{x}}) = E_p\left\{\ln\left(p_{\boldsymbol{\epsilon}}(\mathbf{x};\boldsymbol{\theta}_0,\boldsymbol{\theta}_1)\right) - \ln\left(f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta})\right)\right\} \tag{38}$$

$$\boldsymbol{\theta}_{pt} = \arg\min_{\boldsymbol{\theta}}\left\{D(p_{\mathbf{x}}||f_{\mathbf{x}})\right\} = \arg\min_{\boldsymbol{\theta}}\left\{E_p\left\{-\ln\left(f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta})\right)\right\}\right\}, \tag{39}$$

where $E_p\{\cdot\}$ is the expectation with respect to the true model's pdf, and

$$-\ln\left(f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta})\right) = -N\ln(\pi) - 2N\ln(\sigma_n) + \frac{1}{\sigma_n^2}\left\|\mathbf{x} - \alpha\mathbf{a}(\boldsymbol{\eta})\right\|^2. \tag{40}$$

The last term of (40) can be expanded as follows:

$$\begin{aligned}
\left\|\mathbf{x} - \alpha\mathbf{a}(\boldsymbol{\eta})\right\|^2 &= \left\|\mathbf{x} - (\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1) + (\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1) - \alpha\mathbf{a}(\boldsymbol{\eta})\right\|^2 \qquad (41) \\
&= \left\|\mathbf{x} - (\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1)\right\|^2 + \left\|\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1 - \alpha\mathbf{a}(\boldsymbol{\eta})\right\|^2 \\
&+ \left(\mathbf{x} - (\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1)\right)^H\left(\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1 - \alpha\mathbf{a}(\boldsymbol{\eta})\right) \\
&+ \left(\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1 - \alpha\mathbf{a}(\boldsymbol{\eta})\right)^H\left(\mathbf{x} - (\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1)\right) \qquad (42)
\end{aligned}$$

The expectation of the first term of (42) is the noise covariance, which cannot be minimized and the expectations of the last two terms of (42) are null. Consequently, to minimize the expectation of (40) w.r.t. the argument $\boldsymbol{\theta}$, the equation can be simplified as,

$$\arg\min_{\boldsymbol{\theta}} \{E_p\{-\ln(f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta}))\}\} = \arg\min_{\boldsymbol{\theta}} \left\{\|\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1 - \alpha\mathbf{a}(\boldsymbol{\eta})\|^2\right\}. \tag{43}$$

Let $\mathbf{P}_{\mathbf{a}}$ be the orthogonal projector and $\mathbf{P}_{\mathbf{a}}{}^{\perp} = \mathbf{I}_N - \mathbf{P}_{\mathbf{a}}$ with $\mathbf{P}_{\mathbf{a}} = \frac{\mathbf{a}(\boldsymbol{\eta})\mathbf{a}(\boldsymbol{\eta})^H}{\|\mathbf{a}(\boldsymbol{\eta})\|^2}$, which leads to

$$\begin{aligned}
&\|\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1 - \alpha\mathbf{a}(\boldsymbol{\eta})\|^2 = \left\|\left(\mathbf{P}_{\mathbf{a}} + \mathbf{P}_{\mathbf{a}}{}^{\perp}\right)(\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1 - \alpha\mathbf{a}(\boldsymbol{\eta}))\right\|^2 \\
&= \|\mathbf{P}_{\mathbf{a}}(\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1 - \alpha\mathbf{a}(\boldsymbol{\eta}))\|^2 + \left\|\mathbf{P}_{\mathbf{a}}{}^{\perp}(\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1 - \alpha\mathbf{a}(\boldsymbol{\eta}))\right\|^2 \\
&= \left\|\mathbf{a}(\boldsymbol{\eta})\left(\frac{\mathbf{a}(\boldsymbol{\eta})^H}{\|\mathbf{a}(\boldsymbol{\eta})\|^2}(\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1) - \alpha\right)\right\|^2 + \left\|\mathbf{P}_{\mathbf{a}}{}^{\perp}(\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1)\right\|^2,
\end{aligned}$$

then the parameters that minimize the KLD are,

$$\boldsymbol{\theta}_{pt} = \arg\min_{\boldsymbol{\theta}} \left\{\|\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1 - \alpha\mathbf{a}(\boldsymbol{\eta})\|^2\right\} \Leftrightarrow \begin{cases} \boldsymbol{\eta}_{pt} = \arg\max_{\boldsymbol{\eta}} \left\{\left\|\mathbf{P}_{\mathbf{a}}{}^{\perp}(\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1)\right\|^2\right\} \\ \alpha_{pt} = \frac{\mathbf{a}(\boldsymbol{\eta}_{pt})^H}{\|\mathbf{a}(\boldsymbol{\eta}_{pt})\|^2}(\alpha_0\mathbf{a}_0 + \alpha_1\mathbf{a}_1) \end{cases} \tag{44}$$

with $\alpha_{pt} = \rho_{pt}e^{j\Phi_{pt}}$. This result shows that minimizing the KLD between the true and the misspecified distribution is equivalent to performing misspecified maximum likelihood estimation.