Technical Note
Details on Impulse Response Estimation and Size Determination

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May 3, 2023

1 Derivation of Convolution Terms Using the Fourier Transform Properties

Let $s(t)$ be a band-limited signal of bandwidth smaller than B sampled at frequency $F_s = 1/T_s$ over $N = N_2 - N_1$ samples. The aim of this note is to provide details on the evaluation of the following integral terms, for $(p,q) \in [1,P]^2$,

$$\left[\mathbf{W}_1^\delta\right]_{p,q} = \int_{\mathbb{R}} s(t-\tau_p)s(t-\tau_q)^* \mathrm{d}t \tag{1}$$

$$\left[\mathbf{W}_2^\delta\right]_{p,q} = \int_{\mathbb{R}} (t-\tau_p)s(t-\tau_p)s(t-\tau_q)^* \mathrm{d}t \tag{2}$$

$$\left[\mathbf{W}_3^\delta\right]_{p,q} = \int_{\mathbb{R}} s_p^{(1)}(t-\tau_p)s(t-\tau_q)^* \mathrm{d}t \tag{3}$$

$$\left[\mathbf{W}_4^\delta\right]_{p,q} = \int_{\mathbb{R}} (t-\tau_q)s^{(1)}(t-\tau_p)s(t-\tau_q)^* \mathrm{d}t \tag{4}$$

$$\left[\mathbf{W}_{2,2}^\delta\right]_{p,q} = \int_{\mathbb{R}} (t-\tau_p)(t-\tau_q)s(t-\tau_p)s(t-\tau_q) \mathrm{d}t \tag{5}$$

$$\left[\mathbf{W}_{3,3}^\delta\right]_{p,q} = \int_{\mathbb{R}} s^{(1)}(t-\tau_p)s^{(1)}(t-\tau_q) \mathrm{d}t \tag{6}$$

where $\tau_p = \tau + (p-1)T_s$, $\tau_q = \tau + (q-1)T_s$ and τ are time delays, f_c is carrier frequency and the superscript $^{(1)}$ refers to the first time derivative of signal $s(t)$.

1.1 Prior Considerations

First the Fourier transform of a set of functions are to be evaluated. Remembering that the signal is band-limited of band $B \leq F_s$, one has:

$$s(t) \rightleftharpoons \mathrm{FT}\left\{s(t)\right\}(f) \triangleq S(f) = \left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} s(nT_s)e^{-j2\pi f n T_s}\right) 1_{\left[-\frac{F_s}{2};\frac{F_s}{2}\right]} \tag{7}$$

A first expression is a simple application of the time shift relation when using the Fourier transform of a delayed signal:

$$(t-\tau)s(t-\tau) = ts(t-\tau) - \tau s(t-\tau) \tag{8}$$

Then,

$$\begin{aligned}\mathbf{FT}\left\{(t-\tau)s(t-\tau)\right\} &= \frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f)e^{-j2\pi f\tau}\right) - \tau S(f)e^{-j2\pi f\tau} \\ &= \frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}(S(f))e^{-j2\pi f\tau}\end{aligned} \tag{9}$$

Besides, with the superscript $^{(1)}$ referring to the first time derivative,

$$\mathbf{FT}\left\{s^{(1)}(t-\tau)\right\} = j2\pi f S(f)e^{-j2\pi f\tau} \tag{10}$$

1.2 Coefficients of $\mathbf{W}^{\delta}$

1.2.1 Matrix $\mathbf{W}_1^{\delta}$

$$\begin{aligned}\left[\mathbf{W}_1^{\delta}\right]_{p,q} &= \int_{\mathbb{R}} s(t-\tau_p)s(t-\tau_q)^*\mathrm{d}t \\ &= \int_{\mathbb{R}} s(u-(p-q)T_s)s(u)^*\mathrm{d}u \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} S(f)e^{-j2\pi f(p-q)T_s}S(f)^*\mathrm{d}f\,,\end{aligned}$$

and, using the sum definition of the Fourier transform (7) as a matrices product,

$$\left[\mathbf{W}_1^{\delta}\right]_{p,q} = \frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\mathbf{s}^T\boldsymbol{\nu}(f)^*\right)e^{-j2\pi f(p-q)}\left(\mathbf{s}^H\boldsymbol{\nu}(f)\right)\mathrm{d}f\,.$$

Hence

$$\boxed{\left[\mathbf{W}_1^{\delta}\right]_{p,q} = \frac{1}{F_s}\mathbf{s}^H\mathbf{V}^{\Delta,0}(p-q)\mathbf{s}} \tag{11}$$

with

$$\mathbf{s} = \begin{pmatrix} \dots & s(nT_s) & \dots \end{pmatrix}^T_{N_1\le n\le N_2} \tag{12}$$

$$\boldsymbol{\nu}(f) = \begin{pmatrix} \dots & e^{j2\pi fn} & \dots \end{pmatrix}^T_{N_1\le n\le N_2} \tag{13}$$

$$\mathbf{V}^{\Delta,0}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}}\boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f)e^{-j2\pi fn}\mathrm{d}f \tag{14}$$

$$\left[\mathbf{V}^{\Delta,0}(n)\right]_{k,l} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi f(k-l-n)}\mathrm{d}f = \mathrm{sinc}(k-l-n) = \begin{cases}1 & k-l=n,\\ 0 & \text{else}\end{cases} \tag{15}$$

1.2.2 Matrix $\mathbf{W}_2^\delta$

$$\begin{aligned}
\left[\mathbf{W}_2^\delta\right]_{p,q} &= \int_{\mathbb{R}} (t-\tau_p)s(t-\tau_p)s(t-\tau_q)^*\mathrm{d}t \\
&= \int_{\mathbb{R}} \underbrace{(u-(p-q)T_s)s(u-(p-q)T_s)}_{(9)} s(u)^*\mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f)\right)e^{-j2\pi f p-q)T_s}S(f)^*\mathrm{d}f \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T\mathbf{D}\boldsymbol{\nu}(f)^*\right)e^{-j2\pi f(p-q)}\left(\mathbf{s}^H\boldsymbol{\nu}(f)\right)\mathrm{d}f \\
&= \frac{1}{F_s^2}\mathbf{s}^H\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f)e^{-j2\pi f(p-q)}\mathrm{d}f\right)\mathbf{D}\mathbf{s}\,.
\end{aligned}$$

Hence

$$\boxed{\left[\mathbf{W}_2^\delta\right]_{p,q} = \frac{1}{F_s^2}\mathbf{s}^H\mathbf{V}^{\Delta,0}(p-q)\mathbf{D}\mathbf{s}} \tag{16}$$

with $\mathbf{V}^{\Delta,0}$ defined in (14) and

$$\mathbf{D} = \begin{pmatrix} \dots & n & \dots \end{pmatrix}^T_{N_1\leq n\leq N_2} \tag{17}$$

1.2.3 Matrix $\mathbf{W}_3^\delta$

$$\begin{aligned}
\left[\mathbf{W}_3^\delta\right]_{p,q} &= \int_{\mathbb{R}} s^{(1)}(t-\tau_p)s(t-\tau_q)^*\mathrm{d}t \\
&= \int_{\mathbb{R}} \underbrace{s^{(1)}(u-(p-q)T_s)}_{(10)} s(u)^*\mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} j2\pi f S(f)e^{-j2\pi f(p-q)T_s}S(f)^*\mathrm{d}f \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} j2\pi f\left(\mathbf{s}^T\boldsymbol{\nu}(f)^*\right)e^{-j2\pi f(p-q)}\left(\mathbf{s}^H\boldsymbol{\nu}(f)\right)\mathrm{d}f \\
&= \mathbf{s}^H\left(j2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} f\boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f)e^{-j2\pi f(p-q)}\mathrm{d}f\right)\mathbf{s}\,.
\end{aligned}$$

Hence

$$\boxed{\left[\mathbf{W}_3^\delta\right]_{p,q} = \mathbf{s}^H\mathbf{V}^{\Delta,1}(p-q)\mathbf{s}} \tag{18}$$

with

$$\mathbf{V}^{\Delta,1}(n) = j2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} f\boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f)e^{-j2\pi f n}\mathrm{d}f \tag{19}$$

and

$$\left[\mathbf{V}^{\Delta,1}(n)\right]_{k,l} = j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f e^{j2\pi f(k-l-n)} \mathrm{d}f$$
$$= \frac{1}{k-l-n}\left(\cos\left(\pi(k-l-n)\right) - \mathrm{sinc}\left(k-l-n\right)\right)$$
$$= \begin{cases} 0 & \text{if } k-l=n, \\ (-1)^{|k-l-n|}/(k-l-n) & \text{else} \end{cases} \tag{20}$$

1.2.4 Matrix $\mathbf{W}_4^\delta$

$$\left[\mathbf{W}_4^\delta\right]_{p,q} = \int_{\mathbb{R}} (t-\tau_q) s^{(1)}(t-\tau_p) s(t-\tau_q)^* \mathrm{d}t$$
$$= \int_{\mathbb{R}} \underbrace{s^{(1)}(u-(p-q)T_s)}_{(10)} \left(u s(u)\right)^* \mathrm{d}u$$
$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} j2\pi f S(f) e^{-j2\pi f(p-q)T_s} \left(\frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f)\right)\right)^* \mathrm{d}f$$
$$= \frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} j2\pi f \left(\mathbf{s}^T \boldsymbol{\nu}(f)^*\right) e^{-j2\pi f(p-q)} \left(\mathbf{s}^H \mathbf{D}\boldsymbol{\nu}(f)\right) \mathrm{d}f$$
$$= \frac{1}{F_s}\mathbf{s}^H \mathbf{D}\left(j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f) e^{-j2\pi f(p-q)} \mathrm{d}f\right)\mathbf{s}\,.$$

Hence

$$\boxed{\left[\mathbf{W}_4^\delta\right]_{p,q} = \frac{1}{F_s}\mathbf{s}^H \mathbf{D}\mathbf{V}^{\Delta,1}(p-q)\mathbf{s}} \tag{21}$$

with $\mathbf{D}$ and $\mathbf{V}^{\Delta,1}$ defined in (17) and (19) respectively.

1.2.5 Matrix $\mathbf{W}_{2,2}^\delta$

$$\left[\mathbf{W}_{2,2}^\delta\right]_{p,q} = \int_{\mathbb{R}} (t-\tau_p)(t-\tau_q) s(t-\tau_p) s(t-\tau_q) \mathrm{d}t$$
$$= \int_{\mathbb{R}} \underbrace{(u-(p-q)T_s) s(u-\Delta\tau)}_{(9)} \left(u s(u)\right)^* \mathrm{d}u$$
$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f)\right) e^{-j2\pi f(p-q)T_s} \left(\frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f)\right)\right)^* \mathrm{d}f$$
$$= \frac{1}{F_s^3}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{D}\boldsymbol{\nu}(f)^*\right) e^{-j2\pi f(p-q)} \left(\mathbf{s}^H \mathbf{D}\boldsymbol{\nu}(f)\right) \mathrm{d}f$$
$$= \mathbf{s}^H \mathbf{D}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f) e^{-j2\pi f(p-q)} \mathrm{d}f\right)\mathbf{D}\mathbf{s}\,.$$

Hence

$$\boxed{\left[\mathbf{W}^{\delta}_{2,2}\right]_{p,q} = \mathbf{s}^H \mathbf{D}\mathbf{V}^{\Delta,0}(p-q)\mathbf{D}\mathbf{s}} \tag{22}$$

with $\mathbf{V}^{\Delta,0}$ and $\mathbf{D}$ defined in (14) and (17) respectively.

1.2.6 Matrix $\mathbf{W}^{\delta}_{3,3}$

$$\begin{aligned}
\left[\mathbf{W}^{\delta}_{3,3}\right]_{p,q} &= \int_{\mathbb{R}} s^{(1)}(t-\tau_p)s^{(1)}(t-\tau_q)\mathrm{d}t \\
&= \int_{\mathbb{R}} \underbrace{s^{(1)}(u-(p-q)T_s)}_{(10)} s^{(1)}(u)^*\mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(j2\pi f S(f)e^{-j2\pi f(p-q)T_s}\right)\left(j2\pi f S(f)\right)^* \mathrm{d}f \\
&= F_s \int_{-\frac{1}{2}}^{\frac{1}{2}} 4\pi^2 f^2 \left(\mathbf{s}^T \boldsymbol{\nu}(f)^*\right) e^{-j2\pi f(p-q)} \left(\mathbf{s}^H \boldsymbol{\nu}(f)\right) \mathrm{d}f \\
&= F_s \mathbf{s}^H \left(4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 \boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f) e^{-j2\pi f(p-q)}\mathrm{d}f\right)\mathbf{s}\,.
\end{aligned}$$

Hence

$$\boxed{\left[\mathbf{W}^{\delta}_{3,3}\right]_{p,q} = F_s \mathbf{s}^H \mathbf{V}^{\Delta,2}(p-q)\mathbf{s}} \tag{23}$$

with

$$\mathbf{V}^{\Delta,2}(n) = 4\pi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 \boldsymbol{\nu}(f)\boldsymbol{\nu}^H(f) e^{-j2\pi f n}\mathrm{d}f \tag{24}$$

and

$$\begin{aligned}
\left[\mathbf{V}^{\Delta,2}(n)\right]_{k,l} &= \pi^2 \mathrm{sinc}\,(k-l-n) \\
&\quad + 2\frac{\cos\left(\pi(k-l-n)\right) - \mathrm{sinc}\,(k-l-n)}{(k-l-n)^2}. \qquad (25) \\
&= \begin{cases} \pi^2/3 & \text{if } k-l=n, \\ (-1)^{|k-l-n|} 2/(k-l-n)^2 & \text{else} \end{cases} \qquad (26)
\end{aligned}$$

2 Details on Orthogonal Projectors Upon Subspaces of a Vector Subspace

Let $\mathbf{A}_M = [\ldots, \mathbf{a}_m, \ldots]$ for $m \in [1, M]$ a full-rank matrix of M vectors. The projector upon the vector subspace defined by the column of $\mathbf{A}_M$ is defined by $\mathbf{P}_{\mathbf{A}_M} = \mathbf{A}_M \left(\mathbf{A}_M^H \mathbf{A}_M\right)^{-1} \mathbf{A}_M^H$. Considering $\mathbf{A}_M = [\mathbf{A}_{M-1}, \mathbf{a}_m]$ where $\mathbf{A}_{M-1}$ is the matrix $\mathbf{A}_M$ without the m-th column, the aim of the following developments is to decompose this projector into two projectors: one over $\mathbf{A}_{M-1}$ and the other over $\mathbf{a}_m$. A first approach is to simply separate the two components:

$$\mathbf{P}_{\mathbf{A}_M} = [\mathbf{A}_{M-1}, \mathbf{a}_m]\left([\mathbf{A}_{M-1}, \mathbf{a}_m]^H [\mathbf{A}_{M-1}, \mathbf{a}_m]\right)^{-1} [\mathbf{A}_{M-1}, \mathbf{a}_m]^H \tag{27}$$

Developing the inverse term,

$$\left([\mathbf{A}_{M-1},\mathbf{a}_m]^H[\mathbf{A}_{M-1},\mathbf{a}_m]\right)^{-1}=\begin{bmatrix}\mathbf{A}_{M-1}^H\mathbf{A}_{M-1} & \mathbf{A}_{M-1}^H\mathbf{a}_m\\ \mathbf{a}_m^H\mathbf{A}_{M-1} & \mathbf{a}_m^H\mathbf{a}_m\end{bmatrix}^{-1}=\begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12}\\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{bmatrix} \tag{28}$$

By resorting to the block matrix inversion lemma [1, Sec. 9.1],

$$\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}^{-1}=\begin{bmatrix}\left(\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\left(\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right)^{-1}\\ -\left(\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right)^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \left(\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right)^{-1}\end{bmatrix} \tag{29}$$

one gets the submatrices, defined in (28):

$$\begin{aligned}\mathbf{B}_{11}&=\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}-\mathbf{A}_{M-1}^H\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\mathbf{A}_{M-1}\right)^{-1}\\&=\left(\mathbf{A}_{M-1}^H\left(\mathbf{I}-\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\right)\mathbf{A}_{M-1}\right)^{-1}\\&=\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\right)^{-1} &(30)\\ \mathbf{B}_{21}&=-\left(\mathbf{a}_m^H\mathbf{a}_m-\mathbf{a}_m^H\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\\&=-\left(\mathbf{a}_m^H\left(\mathbf{I}-\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\right)\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\\&=-\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^{\perp}\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1} &(31)\\ \mathbf{B}_{12}&=-\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{a}_m-\mathbf{a}_m^H\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\right)^{-1} &(32)\\ \mathbf{B}_{22}&=\left(\mathbf{a}_m^H\mathbf{a}_m-\mathbf{a}_m^H\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\right)^{-1}\\&=\left(\mathbf{a}_m^H\left(\mathbf{I}-\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\right)\mathbf{a}_m\right)^{-1}\\&=\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^{\perp}\mathbf{a}_m\right)^{-1} &(33)\end{aligned}$$

Using the PosDef identity [1, eq. (185)] for $\mathbf{P}$ and $\mathbf{R}$ invertible, definite positive matrices and $\mathbf{B}$:

$$\begin{aligned}\left(\mathbf{P}^{-1}+\mathbf{B}^H\mathbf{R}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}^H\mathbf{R}^{-1}&=\mathbf{P}\mathbf{B}^H\left(\mathbf{B}\mathbf{P}\mathbf{B}^H+\mathbf{R}\right)^{-1} &(34)\\ \Leftrightarrow-\left(\mathbf{P}^{-1}-\mathbf{B}^H\mathbf{R}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}^H\mathbf{R}^{-1}&=-\mathbf{P}\mathbf{B}^H\left(\mathbf{R}-\mathbf{B}\mathbf{P}\mathbf{B}^H\right)^{-1}, &(35)\end{aligned}$$

for $\mathbf{P}=\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}$, $\mathbf{R}=\mathbf{a}_m^H\mathbf{a}_m$ and $\mathbf{B}=\mathbf{a}_m^H\mathbf{A}_{M-1}$, (35) allows to rewrite $\mathbf{B}_{12}$ as:

$$\mathbf{B}_{12}=-\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{a}_m\right)^{-1} \tag{36}$$

Hence, the computation goes on,

$$\begin{aligned}&\left([\mathbf{A}_{M-1},\mathbf{a}_m]^H[\mathbf{A}_{M-1},\mathbf{a}_m]\right)^{-1}[\mathbf{A}_{M-1},\mathbf{a}_m]^H &(37)\\ &=\begin{bmatrix}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H-\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\\ -\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^{\perp}\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H+\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^{\perp}\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\end{bmatrix} &(38)\end{aligned}$$

and

$$\mathbf{P}_{\mathbf{A}_M} = \mathbf{A}_{M-1}\left(\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H - \left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\right)$$
$$+\,\mathbf{a}_m\left(-\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H + \left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\right), \tag{39}$$

that is, $\mathbf{P}_{\mathbf{A}_M} = \tilde{\mathbf{P}}_{\mathbf{A}_{M-1}} + \tilde{\mathbf{P}}_{\mathbf{a}_m}$ where,

$$\begin{aligned}
\tilde{\mathbf{P}}_{\mathbf{A}_{M-1}} &= \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H - \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H \\
&= \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\left(\mathbf{I} - \mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\right) \\
&= \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp && (40) \\
\tilde{\mathbf{P}}_{\mathbf{a}_m} &= -\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H + \mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H \\
&= \mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\left(\mathbf{I} - \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\right) \\
&= \mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp. && (41)
\end{aligned}$$

This decomposition is not orthogonal, one cannot show that $\tilde{\mathbf{P}}_{\mathbf{A}_{M-1}}\tilde{\mathbf{P}}_{\mathbf{a}_m} = \mathbf{0}$. Here, the aim is to obtain a decomposition including $\mathbf{P}_{\mathbf{A}_{M-1}}$, a first step is to project $\tilde{\mathbf{P}}_{\mathbf{a}_m}$ over this subspace:

$$\begin{aligned}
\tilde{\mathbf{P}}_{\mathbf{a}_m} &= \left(\mathbf{P}_{\mathbf{A}_{M-1}} + \mathbf{P}_{\mathbf{A}_{M-1}}^\perp\right)\tilde{\mathbf{P}}_{\mathbf{a}_m} \\
&= \mathbf{P}_{\mathbf{A}_{M-1}}\tilde{\mathbf{P}}_{\mathbf{a}_m} + \mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^{-1}\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp \\
&= \mathbf{P}_{\mathbf{A}_{M-1}}\tilde{\mathbf{P}}_{\mathbf{a}_m} + \left(\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)\left(\left(\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^H\left(\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)\right)^{-1}\left(\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^H \\
&= \mathbf{P}_{\mathbf{A}_{M-1}}\tilde{\mathbf{P}}_{\mathbf{a}_m} + \mathbf{P}_{\left(\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)} && (42)
\end{aligned}$$

Hence, $\mathbf{P}_{\left(\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)}$ is orthogonal to the subspace defined by $\mathbf{A}_{M-1}$, the rest (underbraced in the following expression) should reduce to $\mathbf{P}_{\mathbf{A}_{M-1}}$,

$$\mathbf{P}_{\mathbf{A}_M} = \tilde{\mathbf{P}}_{\mathbf{A}_{M-1}} + \tilde{\mathbf{P}}_{\mathbf{a}_m} = \underbrace{\tilde{\mathbf{P}}_{\mathbf{A}_{M-1}} + \mathbf{P}_{\mathbf{A}_{M-1}}\tilde{\mathbf{P}}_{\mathbf{a}_m}} + \mathbf{P}_{\left(\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)}. \tag{43}$$

One can verifies this:

$$\begin{aligned}
&\tilde{\mathbf{P}}_{\mathbf{A}_{M-1}} + \mathbf{P}_{\mathbf{A}_{M-1}}\tilde{\mathbf{P}}_{\mathbf{a}_m} \\
&= \mathbf{A}_{M-1}\left(\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp + \underbrace{\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\mathbf{a}_m\right)^{-1}}_{(35)}\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\right) \\
&= \mathbf{A}_{M-1}\left(\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp + \overbrace{\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{a}_m\left(\mathbf{a}_m^H\mathbf{a}_m\right)^{-1}}\mathbf{a}_m^H\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\right) \\
&= \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^\perp\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\underbrace{\left(\mathbf{P}_{\mathbf{a}_m}^\perp + \mathbf{P}_{\mathbf{a}_m}\mathbf{P}_{\mathbf{A}_{M-1}}^\perp\right)} && (44)
\end{aligned}$$

This last underbraced term can be written as

$$\begin{aligned}
\mathbf{P}_{\mathbf{a}_m}^{\perp} + \mathbf{P}_{\mathbf{a}_m}\mathbf{P}_{\mathbf{A}_{M-1}}^{\perp} &= \mathbf{I} - \mathbf{P}_{\mathbf{a}_m} + \mathbf{P}_{\mathbf{a}_m}\left(\mathbf{I} - \mathbf{P}_{\mathbf{A}_{M-1}}\right) \\
&= \mathbf{I} - \mathbf{P}_{\mathbf{a}_m}\mathbf{P}_{\mathbf{A}_{M-1}} \\
&= \mathbf{I} - \mathbf{P}_{\mathbf{A}_{M-1}} + \mathbf{P}_{\mathbf{A}_{M-1}} - \mathbf{P}_{\mathbf{a}_m}\mathbf{P}_{\mathbf{A}_{M-1}} \\
&= \mathbf{P}_{\mathbf{A}_{M-1}}^{\perp} + \mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{P}_{\mathbf{A}_{M-1}} \qquad (45)
\end{aligned}$$

which leads to

$$\begin{aligned}
&\tilde{\mathbf{P}}_{\mathbf{A}_{M-1}} + \mathbf{P}_{\mathbf{A}_{M-1}}\tilde{\mathbf{P}}_{\mathbf{a}_m} \\
&= \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\left(\mathbf{P}_{\mathbf{a}_m}^{\perp} + \mathbf{P}_{\mathbf{a}_m}\mathbf{P}_{\mathbf{A}_{M-1}}^{\perp}\right) \\
&= \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\left(\mathbf{P}_{\mathbf{A}_{M-1}}^{\perp} + \mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{P}_{\mathbf{A}_{M-1}}\right) \\
&= \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\right)^{-1}\underbrace{\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{A}_{M-1}}^{\perp}}_{=\mathbf{0}} + \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{P}_{\mathbf{A}_{M-1}} \\
&= \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H\mathbf{P}_{\mathbf{a}_m}^{\perp}\mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H \\
&= \mathbf{A}_{M-1}\left(\mathbf{A}_{M-1}^H\mathbf{A}_{M-1}\right)^{-1}\mathbf{A}_{M-1}^H = \mathbf{P}_{\mathbf{A}_{M-1}} \qquad (46)
\end{aligned}$$

Finally, one gets the desired orthogonal decomposition,

$$\mathbf{P}_{\mathbf{A}_M} = \mathbf{P}_{\mathbf{A}_{M-1}} + \mathbf{P}_{\left(\mathbf{P}_{\mathbf{A}_{M-1}}^{\perp}\mathbf{a}_m\right)} \qquad (47)$$

References

[1] Kaare B. Petersen and Michael S. Pedersen, "The Matrix Cookbook," Tech. Rep., Technical Univ. Denmark, Kongens Lyngby, Denmark, 2012.