# GNSS L5/E5 Maximum Likelihood Synchronization Performance Degradation under DME Interferences

Lorenzo Ortega [1,2], Corentin Lubeigt [1,3], Jordi Vilà-Valls [3], Eric Chaumette [3]

[1]TéSA Toulouse

[2]IPSA Toulouse

[3]ISAE-SUPAERO Toulouse

Wednesday, April 26, 2023

# Background

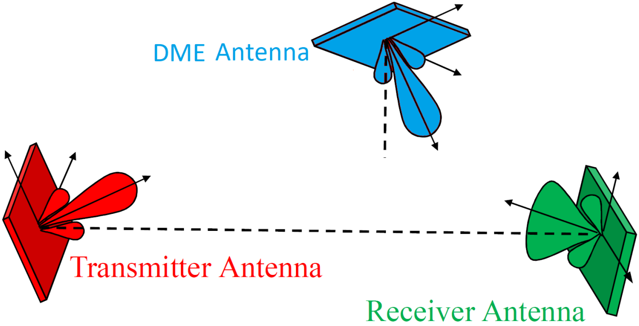


Figure: Illustration of a GNSS system interfered by a DME transmitter.

# Previous Results

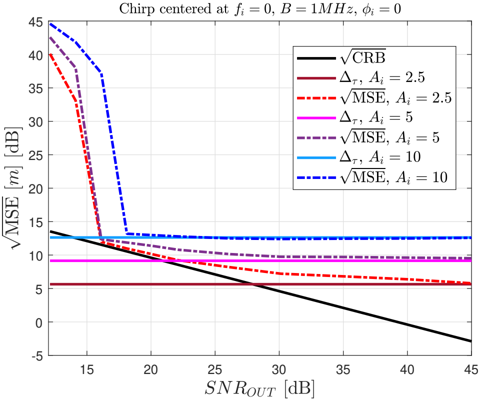


Figure: MSE and bias for the time-delay estimation with a chirp centered at $f_i = 0$, for different jammer amplitudes $A_i$. Chirp bandwidth 1 MHz and initial jammer phase $\phi = 0$.

- Misspecified CRB

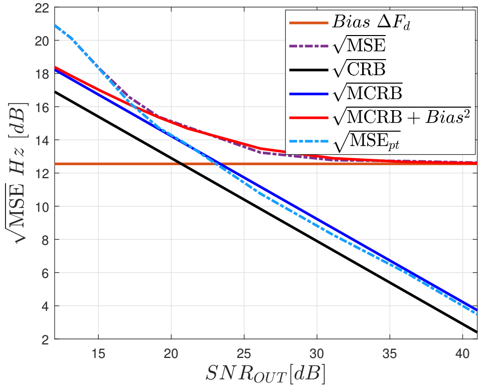


Figure: MSE and bias for the time-delay (left) and Doppler (right) estimation with a chirp centered at $f_i = 0$MHz, for different jammer amplitudes $A_i = 10$. Chirp bandwidth 2 MHz and initial jammer phase $\phi = 0$.

# Content

- Signal Model
  - Correctly Specified Signal Model
  - Misspecified Signal Model
- How to Compute the Bias: Kullback-Leibler Divergence
- How to Compute the MCRB
- MSE Computation and Results for the DME Interference Models

# Signal Model

- **Correctly Specified Signal Model [3]:** A GNSS band-limited signal $s(t)$, with bandwidth $B$, transmitted over a carrier frequency $f_c$ ($\lambda_c = c/f_c$) is considered in this study. The baseband output of the receiver's Hilbert filter can be approximated by,

$$x(t;\boldsymbol{\eta}) = \alpha s\left(t-\tau\right)e^{-j2\pi f_c(b(t-\tau))} + I(t) + n\left(t\right), \quad (1)$$

with $\boldsymbol{\eta} = (\tau, b)^\top$, $I(t)$ an unknown interference, $n(t)$ a complex white Gaussian noise. The discrete vector signal model is built from $N_1 \leq k \leq N_2$ samples at $T_s = 1/F_s$,

$$\mathrm{x} = \alpha \mathrm{a}(\boldsymbol{\eta}) + \mathrm{n} = \rho e^{j\Phi}\mathrm{a}(\boldsymbol{\eta}) + \mathrm{n} = \alpha\boldsymbol{\mu}(\boldsymbol{\eta}) + \mathrm{I} + \mathrm{n}, \quad (2)$$

$$\mathrm{a}(\boldsymbol{\eta}) = \left(\ldots, s(kT_s-\tau)e^{-j2\pi f_c(b(kT_s-\tau)} + \frac{1}{\alpha}I(kT_s)\ldots\right)^\top. \quad (3)$$

The unknown deterministic parameters can be gathered in vector $\boldsymbol{\epsilon}^\top = \left(\sigma_n^2, \rho, \Phi, \boldsymbol{\eta}^\top\right) = \left(\sigma_n^2, \boldsymbol{\theta}^\top\right)$, with $\rho \in \mathbb{R}^+, 0 \leq \Phi \leq 2\pi$. Then, $\mathrm{x} \sim \mathcal{CN}(\alpha\mathrm{a}(\boldsymbol{\eta}), \sigma_n^2 \boldsymbol{I}_N)$.

# Signal Model

- **Misspecified Signal Model [2, 1]:** The misspecified signal model represents the case where the interference is not considered. This nominal case leads to the definition of the misspecified parameter vector $\boldsymbol{\eta}' = [\tau', b']^\top$, and the complete set of unknown parameters $\boldsymbol{\epsilon'}^\top = \left[\sigma_n^2, \rho', \Phi', \boldsymbol{\eta}'^\top\right] = \left[\sigma_n^2, \boldsymbol{\theta}'^\top\right]$, yielding the following signal model at the output of the Hilbert filter,

$$x'(t; \boldsymbol{\eta}') = \alpha' s(t - \tau') e^{-j2\pi f_c b'(t-\tau')} + n(t) \quad (4)$$

with $\alpha' = \rho' e^{j\Phi'}$ and the discrete vector signal model:

$$\mathrm{x}' = \alpha' \boldsymbol{\mu}(\boldsymbol{\eta}') + \mathrm{n} \quad (5)$$

$$\boldsymbol{\mu}(\boldsymbol{\eta}') = (\ldots, s(kT_s - \tau') e^{-j2\pi f_c (b'(kT_s - \tau'))}, \ldots)^\top.$$

The misspecified signal model is represented by a pdf denoted as $\mathrm{x}' \sim \mathcal{CN}(\alpha' \boldsymbol{\mu}(\boldsymbol{\eta}'), \sigma_n^2 \boldsymbol{I}_N)$.

# How to Compute the Bias: Kullback-Leibler Divergence

The pdfs of the Correctly Specified and Misspecified Signal Models are [4]:

$$p_{\epsilon}(\mathrm{x};\epsilon) = \frac{1}{\pi^N \sigma_n^{2N}} e^{\frac{-(\mathrm{x}-\alpha \boldsymbol{a}(\boldsymbol{\eta}))^H (\mathrm{x}-\alpha \boldsymbol{a}(\boldsymbol{\eta}))}{\sigma_n^2}}, \tag{6}$$

$$f_{\epsilon'}(\mathrm{x}';\epsilon') = \frac{1}{\pi^N \sigma_n^{2N}} e^{\frac{-(\mathrm{x}-\alpha' \boldsymbol{\mu}(\boldsymbol{\eta}'))^H (\mathrm{x}-\alpha' \boldsymbol{\mu}(\boldsymbol{\eta}'))}{\sigma_n^2}}. \tag{7}$$

When we consider a misspecified model and the corresponding MMLE, the estimation of the parameters of interest is biased. Those biased estimated parameters are commonly referred to as pseudo-true parameters. We denote them as $\boldsymbol{\theta}_{pt}^{\top} = [\rho_{pt}, \Phi_{pt}, \tau_{pt}, b_{pt}]$.
The pseudo-true parameters are simply those that give the minimum Kullback-Leibler (KLD) Divergence $D(p_{\epsilon}||f_{\epsilon'})$ between the true and assumed models.

$$\Delta\alpha = \alpha_{pt} - \alpha, \quad \boldsymbol{\Delta\eta} = \boldsymbol{\eta}_{pt} - \boldsymbol{\eta}. \tag{8}$$

# How to Compute the MCRB

The MCRB was derived as an extension of the Slepian-Bangs formulas, and it is a combination of two information matrices $\mathrm{A}(\boldsymbol{\theta}_{pt})$ and $\mathrm{B}(\boldsymbol{\theta}_{pt})$

$$\mathrm{MCRB}(\boldsymbol{\theta}_{pt}) = \mathrm{A}(\boldsymbol{\theta}_{pt})^{-1}\mathrm{B}(\boldsymbol{\theta}_{pt})\mathrm{A}(\boldsymbol{\theta}_{pt})^{-1}, \tag{9}$$

where

$$\mathrm{A}(\boldsymbol{\theta}_{pt}) = \frac{2}{\sigma_n^2}\Re\left\{(\delta \mathrm{m})^H\left(\frac{\partial^2 \alpha_{pt}\boldsymbol{\mu}(\boldsymbol{\eta}_{pt})}{\partial\boldsymbol{\theta}_{pt}\partial\boldsymbol{\theta}_{pt}^\top}\right)\right\} - \mathrm{B}(\boldsymbol{\theta}_{pt}),$$

$$\mathrm{B}(\boldsymbol{\theta}_{pt}) = \frac{2}{\sigma_n^2}\Re\left\{\left(\frac{\partial \alpha_{pt}\boldsymbol{\mu}(\boldsymbol{\eta}_{pt})}{\partial\boldsymbol{\theta}_{pt}}\right)^H\left(\frac{\partial \alpha_{pt}\boldsymbol{\mu}(\boldsymbol{\eta}_{pt})}{\partial\boldsymbol{\theta}_{pt}}\right)\right\},$$

$\delta \mathrm{m} \triangleq \alpha \mathrm{a}\left(\boldsymbol{\eta}\right) - \alpha_{pt}\boldsymbol{\mu}(\boldsymbol{\eta}_{pt}) = \alpha\boldsymbol{\mu}\left(\boldsymbol{\eta}\right) + \boldsymbol{I} - \alpha_{pt}\boldsymbol{\mu}(\boldsymbol{\eta}_{pt})$ the mean difference between true and misspecified models.

We have derived **closed-expressions** of these formulas that depend only on the signal and interference samples.

# MSE Computation and Results

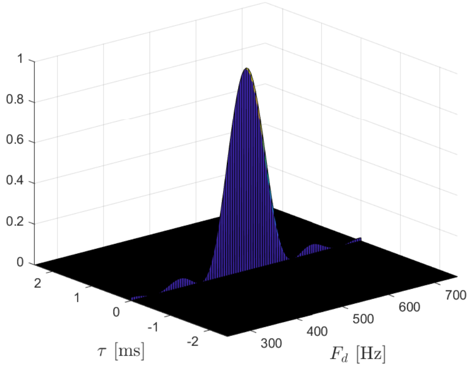


Figure: Ambiguity function of the GPS L5Q signal. The integration time is set to 10 ms, $F_s = 20$ MHz.

# MSE Computation and Results

The following scenario is proposed We consider 7 DME signals interfering the GNSS receiver, with $f_i = [-0.5, -0.25, 0.15, 0, 0.15, 0.25, 0.5]$ MHz, and arriving at the receiver $[1.25, 2.5, 3.75, 5, 6.25, 7.5, 8.75]$ ms after the first chip of the GNSS PRN code.

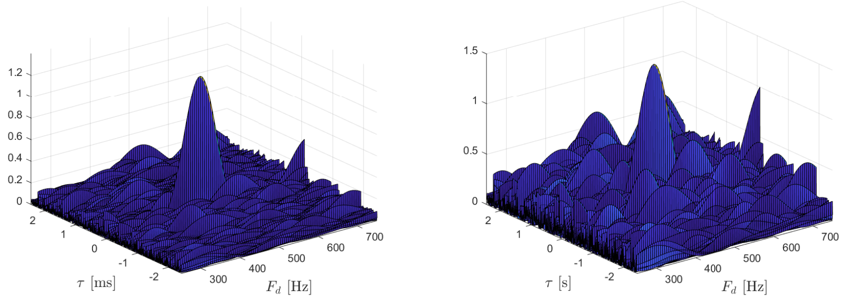


Figure: Ambiguity function of the GPS L5Q signal. The integration time is set to 10 ms, $F_s = 20$ MHz. $I_i = \{30, 32\}$dB.

# MSE Computation and Results

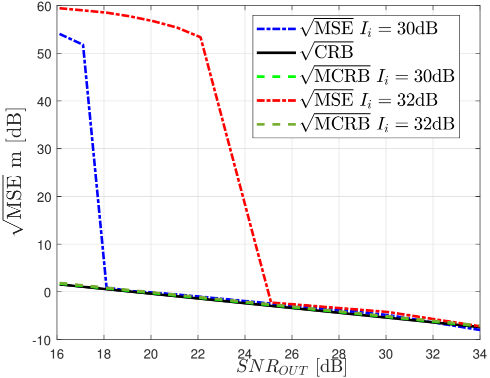


Figure: RMSE of the time-delay of the GPS L5Q signal. The integration time is set to 10 ms, $F_s = 20$ MHz and $I_i = \{30, 32\}$dB.

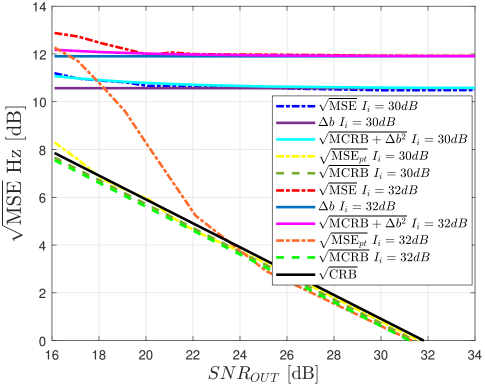


Figure: RMSE of the Doppler of the GPS L5Q signal. The integration time is set to 10 ms, $F_s = 20$ MHz and $I_i = \{30, 32\}$dB.

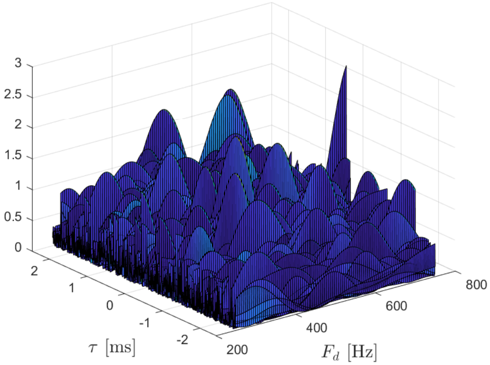


Figure: Ambiguity function of the GPS L5Q signal. The integration time is set to 10 ms, $F_s = 20$ MHz. $I_i = 34$dB.

# References

[1] Corentin Lubeigt et al. "Untangling first and second order statistics contributions in multipath scenarios". In: *Signal Processing* 205 (2023), p. 108868. ISSN: 0165-1684. DOI: https://doi.org/10.1016/j.sigpro.2022.108868.

[2] Hamish McPhee et al. "On the accuracy limits of misspecified delay-Doppler estimation". In: *Signal Processing* 205 (2023), p. 108872. ISSN: 0165-1684. DOI: https://doi.org/10.1016/j.sigpro.2022.108872.

[3] D. Medina et al. "A New Compact CRB for Delay, Doppler and Phase Estimation - Application to GNSS SPP & RTK Performance Characterization". In: *IET Radar, Sonar & Navigation* (2020).

# References

[4] L. Ortega, J. Vilà-Valls, and E. Chaumette. "Theoretical Evaluation of the GNSS Synchronization Performance Degradation under Interferences". In: *Proceedings of the 35th International Technical Meeting of the Satellite Division of The Institute of Navigation (ION GNSS+ 2022)*. 2022, pp. 3758–3767. DOI: https://doi.org/10.33012/2022.18564.


thank you!