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ORIGINAL ARTICLE

On GNSS Synchronization Performance Degradation under Interference Scenarios: Bias and Misspecified Cramér–Rao Bounds

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Abstract

Global navigation satellite systems (GNSSs) play a key role in a plethora of applications, ranging from navigation and timing to Earth observation and space weather characterization. For navigation purposes, interference scenarios are among the most challenging operation conditions, with a clear impact on the maximum likelihood estimates (MLEs) of signal synchronization parameters. While several interference mitigation techniques exist, an approach for theoretically analyzing GNSS MLE performance degradation under interference, which is fundamental for system/receiver design, is lacking. The main goal of this contribution is to provide such analysis, by deriving closed-form expressions of the misspecified Cramér–Rao (MCRB) bound and estimation bias, for a generic GNSS signal corrupted by interference. The proposed bias and MCRB expressions are validated for a linear frequency-modulation chirp signal interference.

Keywords

bias analysis, GNSS synchronization, interference, maximum likelihood, misspecified Cramér-Rao bound

1 | INTRODUCTION

Global navigation satellite systems (GNSSs) (Teunissen & Montenbruck, 2017) appear in a plethora of applications, ranging from navigation and timing to Earth observation, attitude estimation, and space weather characterization. Indeed, reliable position, navigation, and timing information is fundamental in new applications such as intelligent transportation systems and autonomous unmanned ground/air vehicles, for which GNSSs have become the cornerstone source of positioning data, and this dependence can only but grow in the future. However, GNSSs were originally designed to operate in clear sky nominal conditions, and their performance clearly degrades under harsh environments. Among non-nominal operation conditions, multipath, interference (i.e., intentional [jamming] or unintentional), and spoofing conditions are the most challenging, presenting a key issue in safety-critical scenarios (Amin et al., 2016). Interference degrades GNSS performance and can lead to a denial of service or even counterfeit transmissions to control the receiver positioning solution. These effects have been reported in the state of the art, and several interference mitigation countermeasures have been proposed (Amin et al., 2017; Arribas et al., 2019; Borio & Gioia, 2021; Chien, 2015 2018; Fernández-Prades et al., 2016; Liu et al., 2022; Morales-Ferre et al., 2020; Pirayesh & Zeng, 2022).

It is well known that interference impacts the maximum likelihood estimator (MLE) of signal synchronization parameters (i.e., delay, Doppler, phase), which plays a key role in baseband signal processing in standard two-step GNSS receivers (Teunissen & Montenbruck, 2017). While several interference mitigation techniques exist (Morales-Ferre et al., 2020), an approach for theoretically analyzing the GNSS MLE performance degradation induced by an interference (or a set of interferences) is lacking, yet fundamental for system/receiver design. From an estimation perspective, because the system of interest can be formulated as a Gaussian conditional signal model (CSM) under nominal conditions, it is sound to obtain the corresponding Cramér–Rao bound (CRB) (Trees & Bell, 2007). Indeed, the CRB gives an accurate estimation of the mean square error (MSE) of the MLE in the asymptotic region of operation, i.e., in the large sample and/or high signal-to-noise ratio (SNR) regimes of the CSM (Renaux et al., 2006; Stoica & Nehorai, 1990). Even if CRBs for different GNSS receiver architectures under nominal conditions are available in the literature (see (Medina et al., 2020), (Medina et al., 2021), (McPhee et al., 2023a) and references therein), such performance bounds have not been studied for the interference case of interest in this contribution.

The main hypothesis is that the receiver is not aware that an interference is present, and therefore, it assumes that the received signal is only corrupted by additive Gaussian noise as under nominal conditions. This assumption implies that the signal model at the receiver input and the assumed signal model do not coincide, that is, there exists a model mismatch. In this case, the MLE is no longer unbiased, and theoretical characterization leads to closed-form expressions of i) the estimation bias induced by the interference (this result was first presented in Ortega et al. (2022)) and ii) the corresponding misspecified CRB (MCRB) (Richmond & Horowitz, 2015), (Fortunati et al., 2017), (Lubeigt et al., 2023), (McPhee et al., 2023b). The proposed bias and MCRB expressions are validated for a representative linear frequency-modulation (LFM) chirp signal interference. Notably, once a compact MCRB form is derived, this form can be used for i) the derivation of metrics that allow one to compare the robustness of different GNSS signals to interference and to assess the design of new GNSS signals and ii) the design of next-generation interference countermeasures.

2 | TRUE AND MISSPECIFIED SIGNAL MODELS

2.1 | Correctly Specified Signal Model

A GNSS band-limited signal $s(t)$ with bandwidth B is transmitted over a carrier frequency f_c ($\lambda_c = c/f_c, \omega_c = 2\pi f_c$). The synchronization parameters to be estimated are the delay and Doppler shift, $\eta = (\tau, b)^\top$. Under the narrowband assumption, the influence of the Doppler parameter on the baseband signal samples is negligible, $s((1-b)(t-\tau)) \approx s(t-\tau)$ (Dogandzic & Nehorai, 2001). For short observation times, a good approximation of the baseband output of the receiver's Hilbert filter (GNSS signal + interference) is given as follows (Skolnik, 1990):

$$x(t;\eta) = \alpha s(t-\tau)e^{-j2\pi f_c(b(t-\tau))} + I(t) + n(t) \tag{1}$$

where $I(t)$ is a band-limited unknown interference (or set of interferences) within the frequency band of interest, $n(t)$ is complex white Gaussian noise with an unknown variance σ_n^2, and $\alpha = \rho e^{j\Phi}$ is a complex gain. The discrete vector signal model is built from $N = N_1 - N_2 + 1$ samples at $T_s = 1/F_s \leq 1/B$:

$$\mathbf{x} = \alpha \mathbf{a}(\eta) + \mathbf{n} = \rho e^{j\Phi} \mathbf{a}(\eta) + \mathbf{n} = \alpha \mu(\eta) + \mathbf{I} + \mathbf{n} \tag{2}$$

with $\mathbf{x} = (\ldots, x(kT_s), \ldots)^\top$, $\mathbf{I} = (\ldots, I(kT_s), \ldots)^\top$, $\mathbf{n} = (\ldots, n(kT_s), \ldots)^\top$, $N_1 \leq k \leq N_2$ signal samples, and

$$\mathbf{a}(\eta) = (\ldots, s(kT_s - \tau) e^{-j2\pi f_c (b(kT_s - \tau))} + \frac{1}{\alpha} I(kT_s) \ldots)^\top \tag{3}$$

$$\mu(\eta) = (\ldots, s(kT_s - \tau) e^{-j2\pi f_c (b(kT_s - \tau))} \ldots)^\top \tag{4}$$

The unknown deterministic parameters can be gathered in vector $\epsilon^\top = (\sigma_n^2, \rho, \Phi, \eta^\top) = (\sigma_n^2, \theta^\top)$, with $\rho \in \mathbb{R}^+, 0 \leq \Phi \leq 2\pi$. The correctly specified signal model is represented by a probability density function (pdf) denoted as $p_\epsilon(\mathbf{x}; \epsilon)$, which follows a complex circular Gaussian distribution, $\mathbf{x} \sim \mathcal{CN}(\alpha \mathbf{a}(\eta), \sigma_n^2 \mathbf{I}_N)$.

2.2 | Misspecified Signal Model

The misspecified signal model represents the case in which interference is not considered, i.e., when a mismatched MLE (MMLE) is implemented at the receiver. This nominal case leads to the definition of the misspecified parameter vector $\eta' = [\tau', b']^\top$ and the complete set of unknown parameters $\epsilon'^\top = [\sigma_n'^2, \rho', \Phi', \eta'^\top] = [\sigma_n'^2, \theta'^\top]$, yielding the following signal model at the output of the Hilbert filter:

$$x'(t; \eta') = \alpha' s(t - \tau') e^{-j2\pi f_c b'(t - \tau')} + n'(t) \tag{5}$$

where $n'(t)$ is complex white Gaussian noise with an unknown variance $\sigma_n'^2$ and $\alpha' = \rho' e^{j\Phi'}$. Again, we can build the discrete vector signal model from N samples at $T_s = 1/F_s$:

$$\mathbf{x}' = \alpha' \mu(\eta') + \mathbf{n}, \; \mu(\eta') = (\ldots, s(kT_s - \tau') e^{-j2\pi f_c b'(kT_s - \tau')}, \ldots)^\top \tag{6}$$

The misspecified signal model is represented by a pdf denoted as $f_{\epsilon'}(\mathbf{x}; \epsilon')$ that follows a complex circular Gaussian distribution, $\mathbf{x}' \sim \mathcal{CN}(\alpha' \mu(\eta'), \sigma_n'^2 \mathbf{I}_N)$. We then have the following:

$$p_\epsilon(\mathbf{x}; \epsilon) = \frac{1}{\pi^N \sigma_n^{2N}} e^{\frac{-(\mathbf{x} - \alpha \mathbf{a}(\eta))^H (\mathbf{x} - \alpha \mathbf{a}(\eta))}{\sigma_n^2}} \quad f_{\epsilon'}(\mathbf{x}'; \epsilon') = \frac{1}{\pi^N \sigma_n'^{2N}} e^{\frac{-(\mathbf{x} - \alpha' \mu(\eta'))^H (\mathbf{x} - \alpha' \mu(\eta'))}{\sigma_n'^2}} \tag{7}$$

Note that considering the misspecified signal model induces a bias to the corresponding MMLE. These biased estimated parameters are commonly referred to as pseudotrue parameters, $\theta_{pt}^\top = [\rho_{pt}, \Phi_{pt}, \tau_{pt}, b_{pt}]$. For this particular contribution, we are not interested in the noise variance parameter.

3 | MMLE BIAS COMPUTATION VIA KULLBACK–LEIBLER DIVERGENCE

Pseudotrue parameters are simply those that give the minimum Kullback–Leibler divergence (KLD) (Fortunati et al., 2017), $D(p_\epsilon \,||\, f_{\epsilon'}) = E_{p_\epsilon}\left[\ln p_\epsilon(\mathbf{x};\epsilon) - \ln f_{\epsilon'}(\mathbf{x}';\epsilon')\right]$, between the true and assumed models, where $E_{p_\epsilon}[\cdot]$ is the expectation with respect to the true model's pdf:

$$\theta_{pt} = \arg\min_{\theta'}\left\{D(p_\epsilon \,||\, f_{\epsilon'})\right\} = \arg\min_{\theta'}\left\{E_{p_\epsilon}\left[-\ln f_{\epsilon'}(\mathbf{x};\epsilon')\right]\right\} \tag{8}$$

$$E_{p_\epsilon}\left[-\ln f_{\epsilon'}\right] = -N\ln(\pi) - E_{p_\epsilon}\left[2N\ln(\sigma_n')\right] + E_{p_\epsilon}\left[\frac{\left(\mathbf{x}-\alpha\mathbf{a}(\eta)+\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta')\right)^H\left(\mathbf{x}-\alpha\mathbf{a}(\eta)+\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta')\right)}{\sigma_n'^2}\right] \tag{9}$$

We aim to compute the pseudotrue parameters, $\theta_{pt}^\top = \left[\rho_{pt}, \Phi_{pt}, \tau_{pt}, b_{pt}\right]$. We must then minimize Equation (8) with respect to the argument θ', and the equation can be simplified as follows:

$$\begin{aligned}
&\arg\min_{\theta'}\left\{E_{p_\epsilon}\left[-\ln f_{\epsilon'}(\mathbf{x};\epsilon')\right]\right\} \\
&= \arg\min_{\theta'}\left\{E_{p_\epsilon}\left[\begin{array}{c}\frac{1}{\sigma_n'^2}\left[(\mathbf{x}-\alpha\mathbf{a}(\eta))^H(\mathbf{x}-\alpha\mathbf{a}(\eta)) + (\mathbf{x}-\alpha\mathbf{a}(\eta))^H(\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta'))\right. \\ +(\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta'))^H(\mathbf{x}-\alpha\mathbf{a}(\eta)) \\ \left.+(\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta'))^H(\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta'))\right]\end{array}\right]\right\} \\
&= \arg\min_{\theta'}\left\{(\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta'))^H(\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta'))\right\} = \arg\min_{\theta'}\left\{\left\|\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta')\right\|^2\right\}
\end{aligned}$$

We define the orthogonal projector $\Pi_{\mathbf{A}}^\perp = \mathbf{I} - \Pi_{\mathbf{A}}$ with $\Pi_{\mathbf{A}} = \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H$, which leads to the following:

$$\begin{aligned}
\left\|\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta')\right\|^2 &= \left\|\left(\Pi_{\mu(\eta')} + \Pi_{\mu(\eta')}^\perp\right)\left(\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta')\right)\right\|^2 \\
&= \left\|\Pi_{\mu(\eta')}\left(\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta')\right)\right\|^2 + \left\|\Pi_{\mu(\eta')}^\perp\left(\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta')\right)\right\|^2 \\
&= \left\|\Pi_{\mu(\eta')}\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta')\right\|^2 + \left\|\Pi_{\mu(\eta')}^\perp\alpha\mathbf{a}(\eta)\right\|^2 \\
&= \left\|\mu(\eta')\left(\frac{\mu(\eta')^H\alpha\mathbf{a}(\eta)}{\mu(\eta')^H\mu(\eta')} - \alpha'\right)\right\|^2 + \left\|\alpha\mathbf{a}(\eta)\right\|^2 - \left\|\Pi_{\mu(\eta')}\alpha\mathbf{a}(\eta)\right\|^2
\end{aligned}$$

Then, the parameters that minimize the KLD are as follows:

$$\arg\min_{\theta'}\left\{\left\|\alpha\mathbf{a}(\eta)-\alpha'\mu(\eta')\right\|^2\right\} \Leftrightarrow \begin{cases}\alpha_{pt} = \alpha\dfrac{\mu(\eta_{pt})^H\mathbf{a}(\eta)}{\mu(\eta_{pt})^H\mu(\eta_{pt})} \\ \eta_{pt} = \arg\max_{\eta'}\left\{\left\|\Pi_{\mu(\eta')}\alpha\mathbf{a}(\eta)\right\|^2\right\}\end{cases}$$

Here, $\alpha_{pt} = \rho_{pt} e^{j\Phi_{pt}}$ and $\eta_{pt}^{\top} = \left[\tau_{pt}, b_{pt}\right]$. This result may be connected to the asymptotic MMLE behavior (Fortunati et al., 2017):

$$\begin{cases} \widehat{\alpha}' = \dfrac{\mu(\widehat{\eta}')^H \mathbf{x}}{\mu(\widehat{\eta}')^H \mu(\widehat{\eta}')} \\ \widehat{\eta}' = \arg\max_{\eta'} \left\{ \left\| \Pi_{\mu(\eta')} \mathbf{x} \right\|^2 \right\} \end{cases} \underset{SNR \to \infty}{\rightarrow} \begin{cases} \widehat{\alpha}' = \alpha \dfrac{\mu(\widehat{\eta}')^H \mathbf{a}(\eta')}{\mu(\widehat{\eta}')^H \mu(\widehat{\eta}')} = \alpha_{pt} \\ \widehat{\eta}' = \arg\max_{\eta'} \left\{ \left\| \Pi_{\mu(\eta')} \alpha \mathbf{a}(\eta) \right\|^2 \right\} = \eta_{pt} \end{cases} \tag{10}$$

Because the pseudotrue parameters, obtained as the MMLE without noise, are those that give the minimum KLD between the true and assumed models, the bias is defined as $\Delta\alpha = \alpha_{pt} - \alpha$, $\Delta\eta = \eta_{pt} - \eta$.

4 | CLOSED-FORM MCRB EXPRESSIONS FOR A BAND-LIMITED SIGNAL UNDER INTERFERENCE

In Richmond & Horowitz (2015), the MCRB was derived as an extension of the Slepian–Bangs formulas, a result that was later expressed as a combination of two information matrices ($\mathbf{A}(\theta_{pt})$ and $\mathbf{B}(\theta_{pt})$) in Fortunati et al. (2017):

$$\mathbf{MCRB}(\theta_{pt}) = \mathbf{A}(\theta_{pt})^{-1} \mathbf{B}(\theta_{pt}) \mathbf{A}(\theta_{pt})^{-1} \tag{11}$$

with the following relations:

$$\mathbf{A}(\theta_{pt}) = \frac{2}{\sigma_n^2} \Re\left\{ (\delta\mathbf{m})^H \left(\frac{\partial^2 \alpha_{pt} \mu(\eta_{pt})}{\partial\theta_{pt} \partial\theta_{pt}^{\top}} \right) \right\} - \mathbf{B}(\theta_{pt})$$

$$\mathbf{B}(\theta_{pt}) = \frac{2}{\sigma_n^2} \Re\left\{ \left(\frac{\partial \alpha_{pt} \mu(\eta_{pt})}{\partial\theta_{pt}} \right)^H \left(\frac{\partial \alpha_{pt} \mu(\eta_{pt})}{\partial\theta_{pt}} \right) \right\}$$

Here, $\delta\mathbf{m} \triangleq \alpha\mathbf{a}(\eta) - \alpha_{pt}\mu(\eta_{pt}) = \alpha\mu(\eta) + \mathbf{I} - \alpha_{pt}\mu(\eta_{pt})$ is the mean difference between the true and misspecified models.

4.1 | Single-Source Fisher Information Matrix

In $\mathbf{B}(\theta_{pt})$, one can recognize the Fisher information matrix (FIM) of a single-source CSM. A compact expression of this FIM, which depends only on the baseband signal samples, was recently derived in Medina et al. (2020). For completeness, we recall the following:

$$\mathbf{B}(\theta_{pt}) = \frac{2F_s}{\sigma_n^2} \Re\left\{ \mathbf{QWQ}^H \right\} \tag{12}$$

with

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2^* & w_3^* \\ w_2 & W_{2,2} & w_4^* \\ w_3 & w_4 & W_{3,3} \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} j\alpha_{pt}\omega_c b_{pt} & 0 & -\alpha_{pt} \\ 0 & -j\alpha_{pt}\omega_c & 0 \\ e^{j\Phi_{pt}} & 0 & 0 \\ \alpha_{pt} & 0 & 0 \end{bmatrix} \tag{13}$$

Here, the elements of $\mathbf{W}$ can be expressed with respect to the baseband signal samples as follows:

$$w_1 = \frac{1}{F_s}\mathbf{s}^H\mathbf{s},\ w_2 = \frac{1}{F_s^2}\mathbf{s}^H\mathbf{D}\mathbf{s},\ w_3 = \frac{1}{F_s}\mathbf{s}^H\mathbf{V}^{\Delta,1}(0)\mathbf{s},$$
$$w_4 = \frac{1}{F_s}\mathbf{s}^H\mathbf{D}\mathbf{V}^{\Delta,1}(0)\mathbf{s},\ W_{2,2} = \frac{1}{F_s^3}\mathbf{s}^H\mathbf{D}^2\mathbf{s},\ W_{3,3} = F_s\mathbf{s}^H\mathbf{V}^{\Delta,2}(0)\mathbf{s}$$

$\mathbf{s}$, the baseband sample vector, $\mathbf{D}$, $\mathbf{V}^{\Delta,1}(\cdot)$, and $\mathbf{V}^{\Delta,2}(\cdot)$ are defined as follows:

$$\mathbf{s} = \left(\ldots, s(nT_s), \ldots\right)^\top_{N_1 \le n \le N_2} \tag{14a}$$

$$\mathbf{D} = \mathrm{diag}\left(\ldots, n, \ldots\right)_{N_1 \le n \le N_2} \tag{14b}$$

$$\left[\mathbf{V}^{\Delta,1}(q)\right]_{k,l} = \frac{1}{k-l-q}\left(\cos\left(\pi(k-l-q)\right) - \mathrm{sinc}\left(k-l-q\right)\right) \tag{14c}$$

$$\left[\mathbf{V}^{\Delta,2}(q)\right]_{k,l} = \pi^2\mathrm{sinc}\left(k-l-q\right) + 2\frac{cos\left(\pi(k-l-q)\right) - \mathrm{sinc}\left(k-l-q\right)}{\left(k-l-q\right)^2} \tag{14d}$$

We refer the reader to Appendix B for details on the closed-form expressions of $\mathbf{V}^{\Delta,1}(q)$ and $\mathbf{V}^{\Delta,2}(q)$.

4.2 | Model Mismatch Information Matrix

The matrix $\mathbf{A}(\theta_{pt})$ accounts for the model misspecification. Its elements can also be expressed in a compact form as a function of the baseband signal and interference samples as follows:

$$\left[\mathbf{A}(\theta_{pt})\right]_{p,q} = \frac{2F_s}{\sigma_n^2}\Re\left\{\left[\mathbf{Q}_q\right]_{p,.}\mathbf{W}^{\mathbf{A}}\tilde{\alpha}^*\right\} - \left[\mathbf{B}(\theta_{pt})\right]_{p,q}$$
$$\mathbf{W}^{\mathbf{A}} = [\mathbf{w}_1^{\mathbf{A}}\ \mathbf{w}_2^{\mathbf{A}}\ \mathbf{w}_3^{\mathbf{A}}],\ \tilde{\alpha} = \left(\rho e^{j\Phi}, 1, -\rho_{pt}e^{j\Phi_{pt}}\right)^\top \tag{15}$$

with $\mathbf{w}_1^{\mathbf{A}} = [\ldots, w_{1,l}^{\mathbf{A}}, \ldots]^\top$, $\mathbf{w}_2^{\mathbf{A}} = [\ldots, w_{2,l}^{\mathbf{A}}, \ldots]^\top$, and $\mathbf{w}_3^{\mathbf{A}} = [\ldots, w_{3,l}^{\mathbf{A}}, \ldots]^\top$ for $l \in (1, \cdots, 6)$. Here, $\left[\mathbf{Q}_q\right]_{p,.}$ is the p-th row of the matrix $\mathbf{Q}_q$ (refer to Appendix A for $\mathbf{Q}_q$). With $\Delta\tau = \tau - \tau_{pt}$ and $\Delta b = b - b_{pt}$, $\mathbf{W}^{\mathbf{A}}$ is obtained from the following:

$$w_{1,1}^{\mathbf{A}}(\eta)^* = \frac{1}{F_s}\mathbf{s}^H\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,0}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}e^{j\omega_c b\Delta\tau}, \tag{16a}$$

$$w_{1,2}^{\mathbf{A}}(\eta)^* = \frac{1}{F_s^2}\mathbf{s}^H\mathbf{D}\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,0}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}e^{j\omega_c b\Delta\tau} \tag{16b}$$

$$w_{1,3}^{\mathbf{A}}(\eta)^* = \frac{1}{F_s^3}\mathbf{s}^H\mathbf{D}^2\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,0}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}e^{j\omega_c b\Delta\tau} \tag{16c}$$

$$w_{1,4}^{\mathbf{A}}(\eta)^* = \left(-\mathbf{s}^H\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,1}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s} + \frac{j\omega_c\Delta b}{F_s}\mathbf{s}^H\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,0}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}\right).e^{j\omega_c b\Delta\tau} \tag{16d}$$

$$w_{1,5}^{\mathbf{A}}(\eta)^{*}=\left(-\frac{1}{F_s}\mathbf{s}^H\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,0}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}-\frac{1}{F_s}\mathbf{s}^H\mathbf{D}\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,1}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}\right.$$
$$\left.+j\frac{\omega_c\Delta b}{F_s^2}\mathbf{s}^H\mathbf{D}\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,0}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}\right)e^{j\omega_c b\Delta\tau} \tag{16e}$$

$$w_{1,6}^{\mathbf{A}}(\eta)^{*}=\left(-F_s\mathbf{s}^H\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,2}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}-j2\omega_c\Delta b\mathbf{s}^H\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,1}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}\right.$$
$$\left.-\frac{(\omega_c\Delta b)^2}{F_s}\mathbf{s}^H\mathbf{U}\left(\frac{f_c\Delta b}{F_s}\right)\mathbf{V}^{\Delta,0}\left(\frac{\Delta\tau}{T_s}\right)\mathbf{s}\right)e^{j\omega_c b\Delta\tau} \tag{16f}$$

$$w_{2,1}^{\mathbf{A}}=\frac{1}{F_s}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s} \tag{16g}$$

$$w_{2,2}^{\mathbf{A}}=\frac{1}{F_s^2}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{D}\mathbf{s} \tag{16h}$$

$$w_{2,3}^{\mathbf{A}}=\frac{1}{F_s^3}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{D}^2\mathbf{s} \tag{16i}$$

$$w_{2,4}^{\mathbf{A}}=\left(\mathbf{I}^H\mathbf{V}^{\Delta,1}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s}+\frac{j\omega_c b_{pt}}{F_s}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s}\right) \tag{16j}$$

$$w_{2,5}^{\mathbf{A}}=\left(-\frac{1}{F_s}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s}+\frac{1}{F_s}\mathbf{I}^H\mathbf{V}^{\Delta,1}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{D}\mathbf{s}\right.$$
$$\left.+j\frac{\omega_c b_{pt}}{F_s^2}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{D}\mathbf{s}\right) \tag{16k}$$

$$w_{2,6}^{\mathbf{A}}=\left(-F_s\mathbf{I}^H\mathbf{V}^{\Delta,2}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s}+j2\omega_c b_{pt}\mathbf{I}^H\mathbf{V}^{\Delta,1}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s}\right.$$
$$\left.-\frac{(\omega_c b_{pt})^2}{F_s}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s}\right) \tag{16l}$$

$$w_{3,1}^{\mathbf{A}}=w_1,\ w_{3,2}^{\mathbf{A}}=w_2,\ w_{3,3}^{\mathbf{A}}=W_{2,2},\ w_{3,4}^{\mathbf{A}}=w_3,\ w_{3,5}^{\mathbf{A}}=w_4,\ w_{3,6}^{\mathbf{A}}=-W_{3,3} \tag{16m}$$

with

$$\mathbf{U}(p)=\mathrm{diag}\left(\ldots,e^{-j2\pi pn},\ldots\right)_{N_1\leq n\leq N_2} \tag{17}$$

$$\left[\mathbf{V}^{\Delta,0}(q)\right]_{k,l}=\mathrm{sinc}\left(k-l-q\right) \tag{18}$$

Proof. See Appendices A and B. □

4.3 | Implementation of the Bias and MCRB Expressions

In this section, we provide a step-by-step explanation of how to calculate the bias and MCRB of the synchronization parameters of the received signal:

$$x(t;\eta)=\alpha s(t-\tau)e^{-j2\pi f_c(b(t-\tau))}+I(t)+n(t) \tag{19}$$

- First, we must calculate the parameters $\alpha_{pt} = \rho_{pt} e^{j\Phi_{pt}}$ and $\eta_{pt}^{\top} = \left[\tau_{pt}, b_{pt}\right]$ from Equation (10).
- Then, we compute the bias of the synchronization parameters as $\Delta\alpha = \alpha_{pt} - \alpha$, $\Delta\eta = \eta_{pt} - \eta$.
- To compute the MCRB, we first compute the single-source FIM $\mathbf{B}(\theta_{pt})$. This process is described in Section 4.1.
- Then, we compute the model mismatch information matrix $\mathbf{A}(\theta_{pt})$. To do this, we apply the following steps:
  - We compute $\tilde{\alpha}$ from Equation (15).
  - To compute $\mathbf{W}^{\mathbf{A}}$, we define $\Delta\tau = \tau - \tau_{pt}$ and $\Delta b = b - b_{pt}$. Then, we compute the elements of the matrix given by Equations (16a)–(16m).
  - We next compute the matrices $\mathbf{Q}_q$, with $q = \{1, 2, 3, 4\}$, which are included in Appendix A.
  - Finally, we compute $\left[\mathbf{A}(\theta_{pt})\right]_{p,q} = \frac{2F_s}{\sigma_n^2} \Re\left\{\left[\mathbf{Q}_q\right]_{p,.} \mathbf{W}^{\mathbf{A}} \tilde{\alpha}^*\right\} - \left[\mathbf{B}(\theta_{pt})\right]_{p,q}$.
- The MCRB is then computed as $\mathbf{MCRB}(\theta_{pt}) = \mathbf{A}(\theta_{pt})^{-1} \mathbf{B}(\theta_{pt}) \mathbf{A}(\theta_{pt})^{-1}$.

5 | VALIDATION

Let us consider the case in which a global positioning system (GPS) L1 C/A signal experiences interference from a jammer that is generating an LFM chirp signal, which is defined as follows:

$$I(t) = \Pi_T(t) \times e^{j\pi\alpha_c t^2 + j\phi}, \; \Pi_T(t) = \begin{cases} A_i & \text{for } 0 \leq t < T \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

where α_c is the chirp rate, A_i is the amplitude, and $T = NT_s$ is the waveform period. The instantaneous frequency is $f(t) = \frac{1}{2\pi}\frac{d}{dt}\left(\pi\alpha_c t^2\right) = \alpha_c t$, and therefore, the waveform bandwidth is $B = \alpha_c T$. We consider the case in which, after the Hilbert filter, the chirp is located at the baseband frequency, i.e., the central frequency of the chirp is $f_i = 0$. Then, the chirp equation can be rewritten as follows:

$$I(t) = \Pi_T(t) \times e^{j\pi\alpha_c (t - T/2)^2 + j\phi}, \; \Pi_T(t) = \begin{cases} A_i & \text{for } 0 \leq t < T \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

The MSE and bias results for the parameters of interest, $\theta^T = [\rho, \Phi, \eta^T]$, are shown in Figures 1–4, with respect to the SNR at the output of the matched filter (i.e., SNR_{OUT}) and considering the following setup: a GNSS receiver with $F_s = 4$ MHz and a chirp bandwidth equal to 2 MHz, with initial phase $\phi = 0$ and amplitude $A_i = 10$. The number of Monte Carlo iterations is set to 1000. In the results, one can observe that i) the root MSE ($\sqrt{MSE}$) of the true parameter converges to $\sqrt{MCRB + Bias^2}$, ii) $\sqrt{MSE}$ of the pseudotrue parameter converges to $\sqrt{MCRB}$, and iii) $\sqrt{MCRB}$ is always higher than $\sqrt{CRB}$ (refer to (Medina et al., 2020)), which represents the asymptotic estimation performance of the parameters without any source of interference. Such results validate and prove the exactness of the proposed MCRB and bias expressions. Finally, we emphasize that the MCRB characterizes the MMLE asymptotically and is therefore unable to evaluate any occurrences prior to the convergence region. Therefore, the calculation of the MSE of the MMLE also indicates the threshold from which the MCRB theoretically characterizes the MSE of the MMLE.

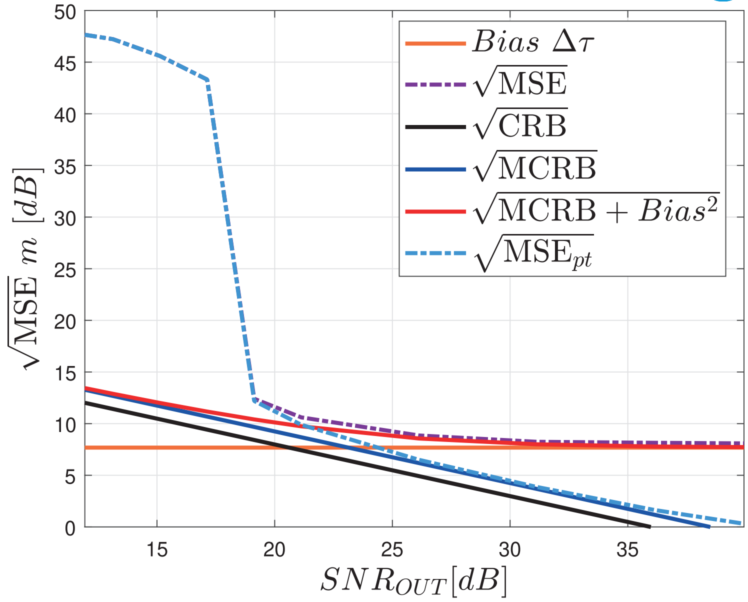

FIGURE 1 MMLE root MSE for the time-delay τ estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a chirp signal with $B = 2$ MHz, $A_i = 10$, and initial phase $\phi = 0$. The integration time is set to 2 ms.

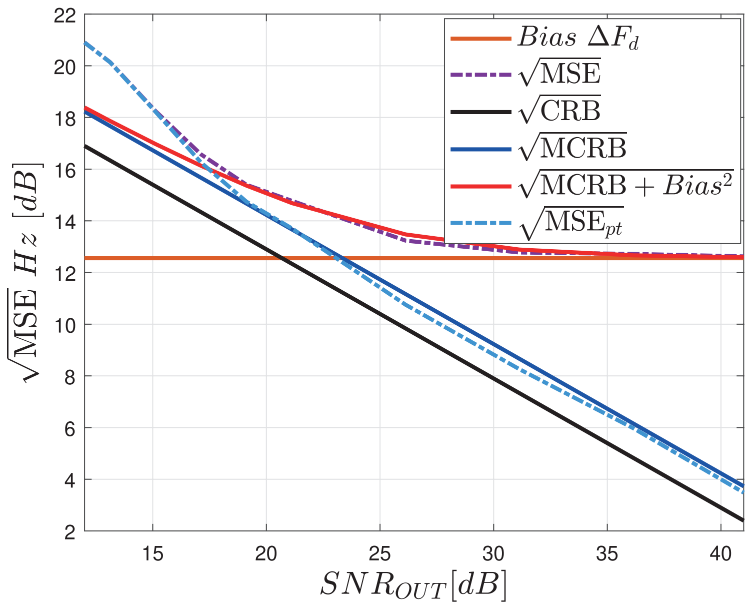

FIGURE 2 MMLE root MSE for the Doppler F_d estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a chirp signal with $B = 2$ MHz, $A_i = 10$, and initial phase $\phi = 0$. The integration time is set to 2 ms.

In a second example, we evaluate the degradation caused by a single tone located at frequency $f_i = 0.5$ MHz. For this particular case, the interference samples are given by $\boldsymbol{I} = I(\cdots, A_i e^{j2\pi f_i k T_s + j\phi}, \cdots)$, which is a complex function, and can be rewritten as follows:

$$\boldsymbol{I} = I(\cdots, A_i e^{j2\pi f_i k T_s + j\phi}, \cdots) = I\left(\cdots, A_i\left(\cos(2\pi f_i k T_s + \phi) + j\sin(2\pi f_i k T_s + \phi)\right), \cdots\right) \tag{22}$$

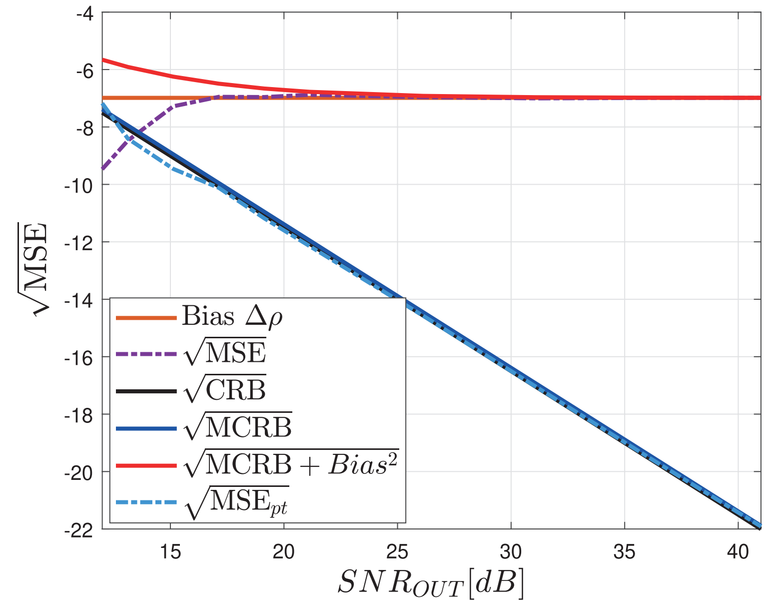

FIGURE 3 MMLE root MSE for the amplitude ρ estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a chirp signal with $B = 2$ MHz, $A_i = 10$, and initial phase $\phi = 0$. The integration time is set to 2 ms.

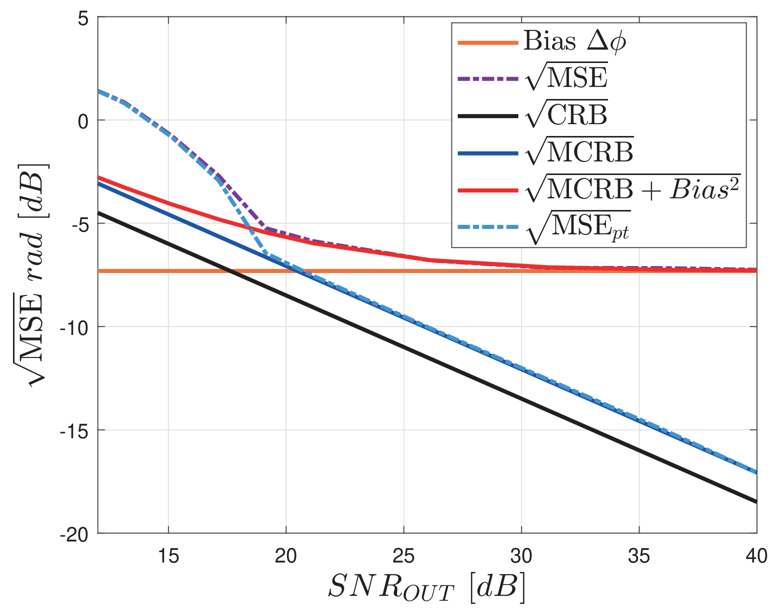

FIGURE 4 MMLE root MSE for the phase Φ estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a chirp signal with $B = 2$ MHz, $A_i = 10$, and initial phase $\phi = 0$. The integration time is set to 2 ms.

where ϕ is the initial phase of the tone and A_i is the amplitude of the tone. For our particular scenario, we set the initial phase to $\pi / 2$ and $A_i = 10$. In Figures 5, 7, 9, and 11, we illustrate the MSE and bias results for the parameters of interest, $\theta^T = [\rho, \Phi, \eta^T]$, as a function of the SNR at the output of the match filter, SNR_{OUT}. We set $F_s = 4$ MHz and the integration time to 2 ms. Note that the MSE converges to the theoretical result, re-validating the closed-form expressions. Moreover, in Figures 6, 8, 10, and 12, we also include one scenario in which the integration time

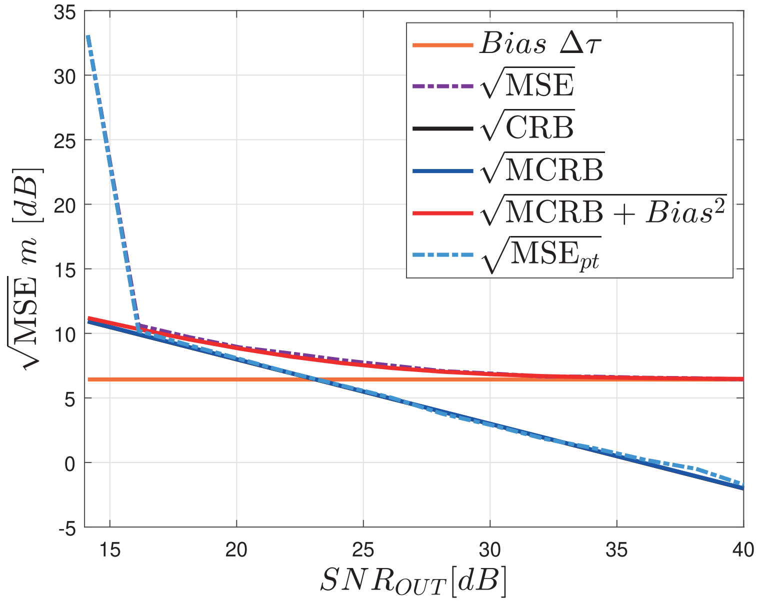

FIGURE 5 MMLE root MSE for the time-delay τ estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a tone signal with $f_i = 0.5$ MHz, $A_i = 10$, and initial phase $\phi = \pi / 2$. The integration time is set to 2 ms.

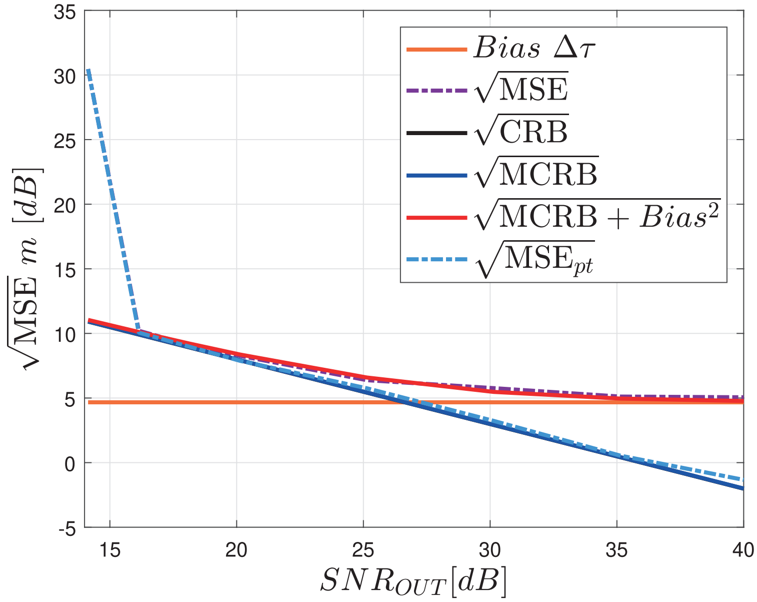

FIGURE 6 MMLE root MSE for the time-delay τ estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a tone signal with $f_i = 0.5$ MHz, $A_i = 10$, and initial phase $\phi = \pi / 2$. The integration time is set to 4 ms.

is set to 4 ms. Note that for this particular case, the bias is lower and the Doppler estimation performance is improved. This result can be proved theoretically owing to the closed-form expressions of the FIM, which allow us to assess how the different design parameters affect the calculation of the MSE of the MLE. For this particular case, increasing the integration time increases the dimension of the matrices $\mathbf{D}$ and $\mathbf{D}^2$, which are related to the Fisher matrix parameters of the Doppler parameter. As the integration time increases, the estimation performance improves.

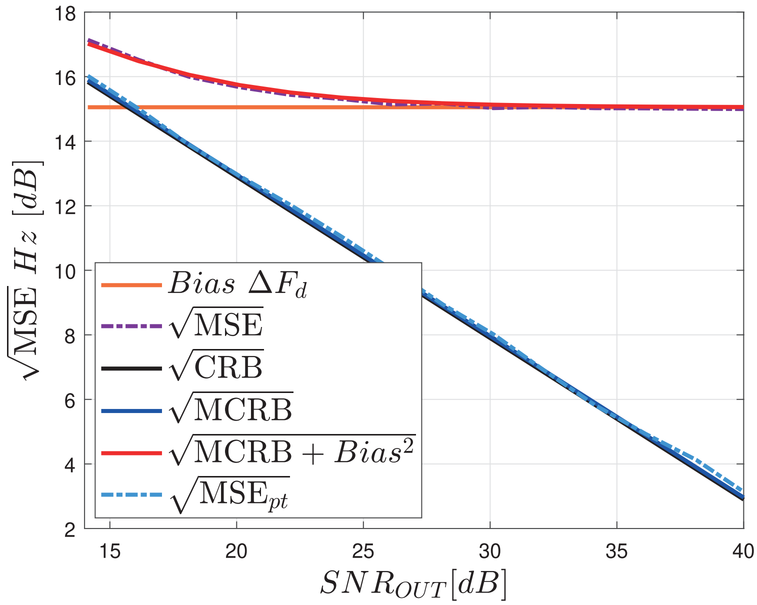

FIGURE 7 MMLE root MSE for the Doppler estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a tone signal with $f_i = 0.5$ MHz, $A_i = 10$, and initial phase $\phi = \pi / 2$. The integration time is set to 2 ms.

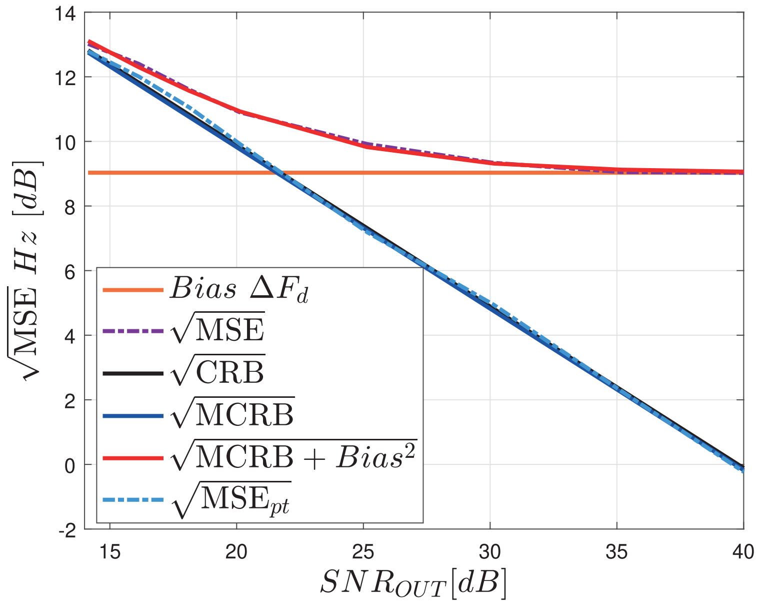

FIGURE 8 MMLE root MSE for the Doppler estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a tone signal with $f_i = 0.5$ MHz, $A_i = 10$, and initial phase $\phi = \pi / 2$. The integration time is set to 4 ms.

6 | CONCLUSION

It is well documented in the literature that interference signals can have a substantial impact on the performance of GNSS receivers, but to the best of the authors' knowledge, from an estimation perspective, an approach for theoretically analyzing the impact of such interference on the first GNSS receiver stage (i.e., time-delay

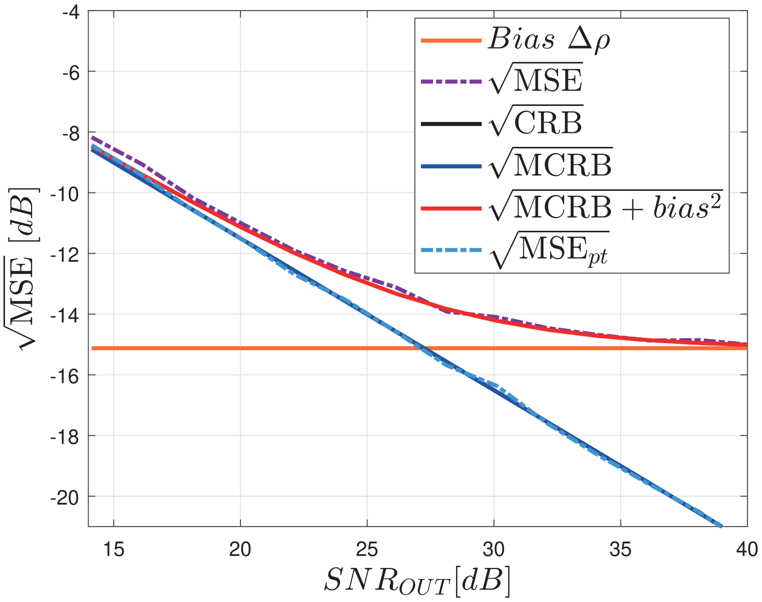

FIGURE 9 MMLE root MSE for the amplitude ρ estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a tone signal with $f_i = 0.5$ MHz, $A_i = 10$, and initial phase $\phi = \pi / 2$. The integration time is set to 2 ms.

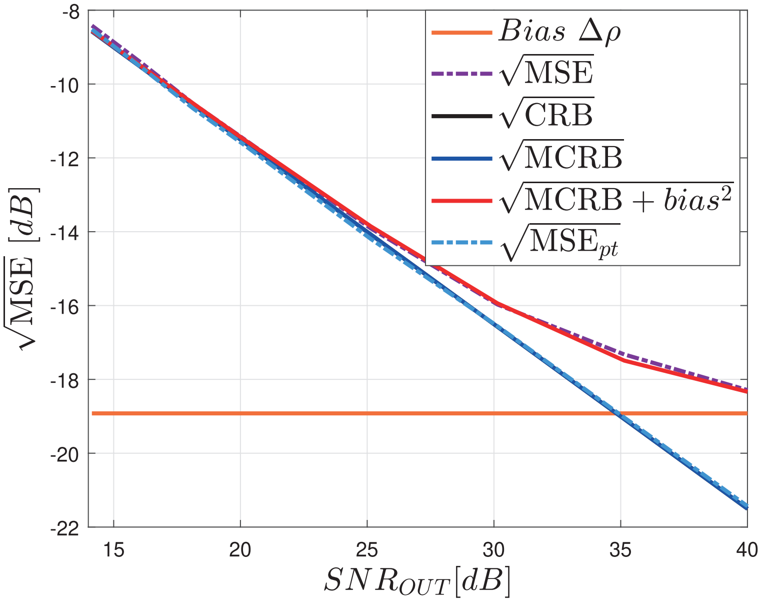

FIGURE 10 MMLE root MSE for the amplitude ρ estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a tone signal with $f_i = 0.5$ MHz, $A_i = 10$, and initial phase $\phi = \pi / 2$. The integration time is set to 4 ms.

and Doppler estimation) is lacking. In practice, at the receiver, there exists a model mismatch, and interference induces both i) an estimation bias and ii) a variance degradation. In this contribution, we provided theoretical closed-form expressions that characterize the MSE for the MLEs of the GNSS synchronization parameters, that is, bias and MCRB. Comparing these results with the standard CRB, associated with the unbiased MLEs without any interference, allows one to theoretically

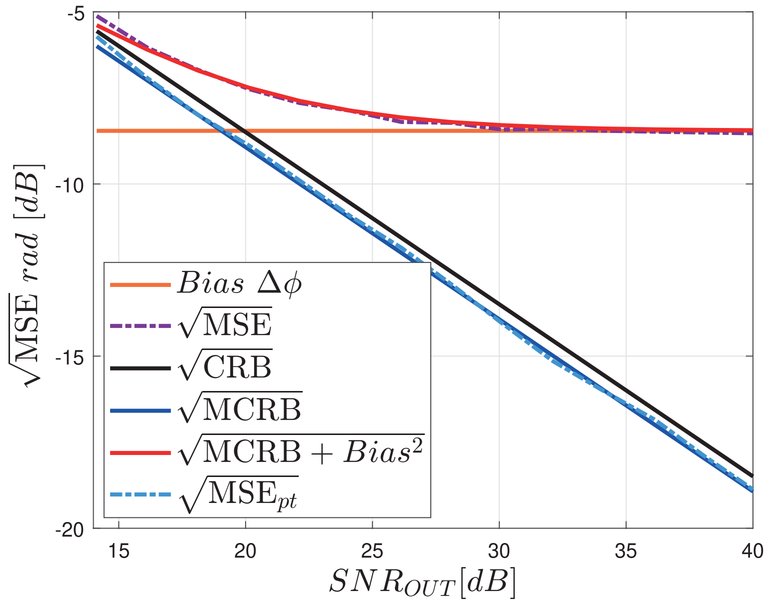

FIGURE 11 MMLE root MSE for the phase Φ estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a tone signal with $f_i = 0.5$ MHz, $A_i = 10$, and initial phase $\phi = \pi / 2$. The integration time is set to 2 ms.

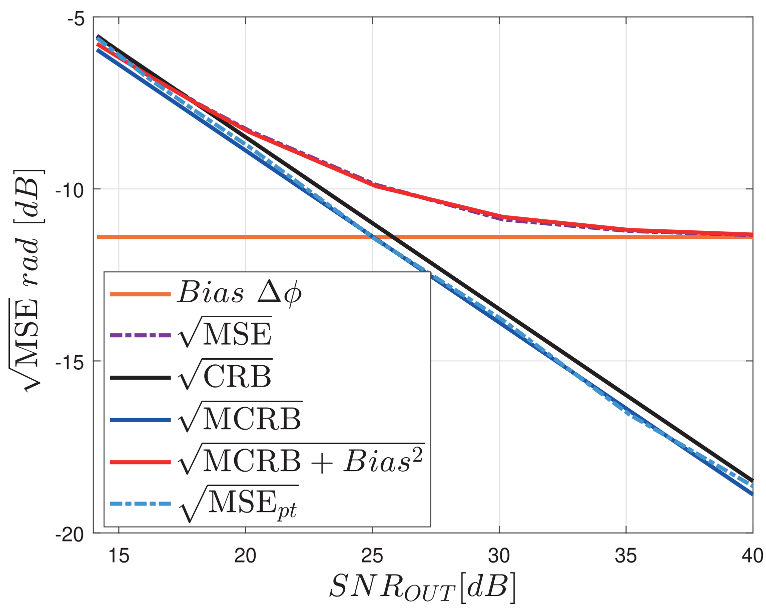

FIGURE 12 MMLE root MSE for the phase Φ estimation with respect to the true and pseudotrue parameters and the corresponding bounds. The interference is a tone signal with $f_i = 0.5$ MHz, $A_i = 10$, and initial phase $\phi = \pi / 2$. The integration time is set to 4 ms.

characterize the performance degradation of the time-delay and Doppler estimation. The exactness of the proposed expressions was validated for a representative case of a chirp interference jamming a GPS L1 C/A signal. Results were provided to demonstrate this validity and the impact on both time-delay and Doppler estimation. Importantly, such analyses may provide a starting point for deriving robustness metrics or new GNSS signals and for designing interference countermeasures. □

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CONFLICT OF INTEREST

The authors declare no potential conflicts of interest.

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APPENDIX

A | ON THE COMPUTATION OF $\mathbf{A}(\boldsymbol{\theta}_{PT})$

To compute $\mathbf{A}(\theta_{pt})$, continuous time expressions are considered: $\mu(t;\eta) = s(t-\tau)e^{-j\omega_c b(t-\tau)}$, $\delta m(t) = \alpha\mu(t;\eta) + I(t) - \alpha_{pt}\mu(t;\eta_{pt}) = \tilde{\mathbf{A}}(t)\tilde{\alpha}$, with $\tilde{\mathbf{A}}(t) = \left[\mu(t;\eta), I(t), \mu(t;\eta_{pt})\right]$ and $\tilde{\alpha} = \left(\rho e^{j\Phi}, 1, -\rho_{pt}e^{j\Phi_{pt}}\right)^\top$, which leads to the discrete expression $\delta\mathbf{m} = \tilde{\mathbf{A}}\tilde{\alpha} = \left[\mu(\eta), \mathbf{I}, \mu(\eta_{pt})\right]\tilde{\alpha}$. The second derivative of interest can be written in matrix form as follows:

$$\frac{\partial^2 \alpha_{pt}\mu(t;\eta_{pt})}{\partial\theta_{pt}\partial\theta_{pt}^\top} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{Q}_3 & \mathbf{Q}_4 \end{bmatrix}\left(\mathcal{D}(t;\eta_{pt}) \otimes \mathbf{I}_4\right)e^{-j\omega_c b_{pt}(t-\tau_{pt})} \tag{A1}$$

with

$$\mathbf{Q}_1 = \begin{bmatrix} -\alpha_{pt}\omega_c^2 b_{pt}^2 & 0 & 0 & -j2\alpha_{pt}\omega_c b_{pt} & 0 & \alpha_{pt} \\ j\alpha_{pt}\omega_c & \alpha_{pt}\omega_c^2 b_{pt} & 0 & 0 & j\alpha_{pt}\omega_c & 0 \\ je^{j\Phi_{pt}}\omega_c b_{pt} & 0 & 0 & -e^{j\Phi_{pt}} & 0 & 0 \\ -\alpha_{pt}\omega_c b_{pt} & 0 & 0 & -j\alpha_{pt} & 0 & 0 \end{bmatrix},$$

$$\mathbf{Q}_2 = \begin{bmatrix} j\alpha_{pt}\omega_c & \alpha_{pt}\omega_c^2 b_{pt} & 0 & 0 & j\alpha_{pt}\omega_c & 0 \\ 0 & 0 & -\alpha_{pt}\omega_c^2 & 0 & 0 & 0 \\ 0 & -je^{j\Phi_{pt}}\omega_c & 0 & 0 & 0 & 0 \\ 0 & \alpha_{pt}\omega_c & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Q}_3 = \begin{bmatrix} je^{j\Phi_{pt}}\omega_c b_{pt} & 0 & 0 & -e^{j\Phi_{pt}} & 0 & 0 \\ 0 & -je^{j\Phi_{pt}}\omega_c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ je^{j\Phi_{pt}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{Q}_4 = \begin{bmatrix} -\alpha_{pt}\omega_c b_{pt} & 0 & 0 & -j\alpha_{pt} & 0 & 0 \\ 0 & \alpha_{pt}\omega_c & 0 & 0 & 0 & 0 \\ je^{j\Phi_{pt}} & 0 & 0 & 0 & 0 & 0 \\ -\alpha_{pt} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{D}(t;\tau_{pt}) = \begin{bmatrix} s(t-\tau_{pt}) \\ (t-\tau_{pt})s(t-\tau_{pt}) \\ (t-\tau_{pt})^2 s(t-\tau_{pt}) \\ s^{(1)}(t-\tau_{pt}) \\ (t-\tau_{pt})s^{(1)}(t-\tau_{pt}) \\ s^{(2)}(t-\tau_{pt}) \end{bmatrix} = \begin{bmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \\ d_4(t) \\ d_5(t) \\ d_6(t)] \end{bmatrix} \tag{A2}$$

where $s^{(1)}(\cdot)$ and $s^{(2)}(\cdot)$ refer to the first and second time derivatives, respectively. The product of the mean difference term and the Hessian matrix, under its discrete form, can be written as follows:

$$\delta\mathbf{m}^H \left[\frac{\partial^2 \alpha_{pt}\mu(\eta_{pt})}{\partial\theta_{pt}\partial\theta_{pt}^T}\right]_{p,q} = \left(\tilde{\mathbf{A}}\tilde{\alpha}\right)^H \left(\left[\mathbf{Q}_q\right]_{p,\cdot}\left[\ldots,\mathcal{D}(kT_s;\eta_{pt})e^{-j\omega_c b_{pt}(kT_s-\tau_{pt})},\ldots\right]_{N_1\le k\le N_2}\right)^\top \tag{A3}$$

This product can also be written as follows:

$$\delta\mathbf{m}^H \left[\frac{\partial^2 \alpha_{pt}\mu(\eta_{pt})}{\partial\theta_{pt}\partial\theta_{pt}^T}\right]_{p,q} = \left[\mathbf{Q}_q\right]_{p,\cdot}\sum_{k=N_1}^{N_2}\begin{bmatrix}\beta_1\\ \vdots \\ \beta_6\end{bmatrix} \tag{A4}$$

with

$$\begin{aligned}\beta_l &= \alpha^*\mu(kT_s;\eta)^* d_l(kT_s)e^{-j\omega_c b_{pt}(kT_s-\tau_{pt})} \\ &\quad + I(kT_s)^* d_l(kT_s)e^{-j\omega_c b_{pt}(kT_s-\tau_{pt})} - \alpha_{pt}^*\mu(kT_s;\eta_{pt})^* d_l(kT_s)e^{-j\omega_c b_{pt}(kT_s-\tau_{pt})}\end{aligned}$$

When the number of samples tends to infinity, each β_l is the sum of three integrals:

$$\begin{aligned}&\lim_{(N_1,N_2)\to(-\infty,+\infty)} T_s\sum_{k=N_1}^{N_2}\mu(kT_s;\eta)^* d_l(kT_s)e^{-j\omega_c b_{pt}(kT_s-\tau_{pt})} \\ &\qquad = \int_{\mathbb{R}}\mu(t;\eta)^* d_l(t)e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt = w_{1,l}^{\mathbf{A}}\end{aligned}$$

$$\begin{aligned}&\lim_{(N_1,N_2)\to(-\infty,+\infty)} T_s\sum_{k=N_1}^{N_2} I(kT_s)^* d_l(kT_s)e^{-j\omega_c b_{pt}(kT_s-\tau_{pt})} \\ &\qquad = \int_{\mathbb{R}} I(t)^* d_l(t)e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt = w_{2,l}^{\mathbf{A}}\end{aligned}$$

$$\lim_{(N_1,N_2)\to(-\infty,+\infty)} T_s \sum_{k=N_1}^{N_2} \mu(kT_s;\eta_{pt})^* d_l(kT_s) e^{-j\omega_c b_{pt}(kT_s-\tau_{pt})}$$
$$= \int_{\mathbb{R}} \mu(t;\eta_{pt})^* d_l(t) e^{-j\omega_c b_{pt}(t-\tau_{pt})} dt = w_{3,l}^{\mathbf{A}} \tag{A5}$$

This result leads to the expression in Equation (15):

$$\lim_{(N_1,N_2)\to(-\infty,+\infty)} \delta\mathbf{m}^H \left[\frac{\partial^2 \alpha_{pt}\mu(\eta_{pt})}{\partial\theta_{pt}\partial\theta_{pt}^T}\right]_{p,q} = F_s\left[\mathbf{Q}_q\right]_{p,\cdot} \begin{bmatrix} w_{1,1}^{\mathbf{A}} & w_{2,1}^{\mathbf{A}} & w_{3,1}^{\mathbf{A}} \\ \vdots & \vdots & \vdots \\ w_{1,6}^{\mathbf{A}} & w_{2,6}^{\mathbf{A}} & w_{3,6}^{\mathbf{A}} \end{bmatrix} \tilde{\alpha}^* \tag{A6}$$

Then, the computation of $\mathbf{A}(\theta_{pt})$ is reduced to three sets of integrals. The first set of integrals is as follows:

$$\begin{aligned}
w_{1,1}^{\mathbf{A}}(\eta) &= \int_{\mathbb{R}} s(t-\tau_{pt})s(t-\tau)^* e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b(t-\tau)\right)}dt, \\
w_{1,2}^{\mathbf{A}}(\eta) &= \int_{\mathbb{R}} (t-\tau_{pt})s(t-\tau_{pt})s(t-\tau)^* e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b(t-\tau)\right)}dt \\
w_{1,3}^{\mathbf{A}}(\eta) &= \int_{\mathbb{R}} (t-\tau_{pt})^2 s(t-\tau_{pt})s(t-\tau)^* e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b(t-\tau)\right)}dt, \\
w_{1,4}^{\mathbf{A}}(\eta) &= \int_{\mathbb{R}} s^{(1)}(t-\tau_{pt})s(t-\tau)^* e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b(t-\tau)\right)}dt \\
w_{1,5}^{\mathbf{A}}(\eta) &= \int_{\mathbb{R}} (t-\tau_{pt})s^{(1)}(t-\tau_{pt})s(t-\tau)^* e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b(t-\tau)\right)}dt, \\
w_{1,6}^{\mathbf{A}}(\eta) &= \int_{\mathbb{R}} s^{(2)}(t-\tau_{pt})s(t-\tau)^* e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b(t-\tau)\right)}dt
\end{aligned} \tag{A7}$$

The corresponding closed-form expressions are given in Equations (16a)–(16f). The derivation of $w_{1,1}^{\mathbf{A}}(\eta)$, $w_{1,2}^{\mathbf{A}}(\eta)$, and $w_{1,4}^{\mathbf{A}}(\eta)$ can be found in Lubeigt et al. (2020) (Equations (A.27), (A.28), and (A.29), respectively). The remaining terms are derived in Appendix B. The second set of integrals is as follows:

$$\begin{aligned}
w_{2,1}^{\mathbf{A}} &= \int_{\mathbb{R}} s(t-\tau_{pt})I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt, \\
w_{2,2}^{\mathbf{A}} &= \int_{\mathbb{R}} (t-\tau_{pt})s(t-\tau_{pt})I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt \\
w_{2,3}^{\mathbf{A}} &= \int_{\mathbb{R}} (t-\tau_{pt})^2 s(t-\tau_{pt})I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt, \\
w_{2,4}^{\mathbf{A}} &= \int_{\mathbb{R}} s^{(1)}(t-\tau_{pt})I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt \\
w_{2,5}^{\mathbf{A}} &= \int_{\mathbb{R}} (t-\tau_{pt})s^{(1)}(t-\tau_{pt})I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt, \\
w_{2,6}^{\mathbf{A}} &= \int_{\mathbb{R}} s^{(2)}(t-\tau_{pt})I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt
\end{aligned} \tag{A8}$$

The corresponding closed-form expressions are given in Equations (16g)–(16l). The derivation of these terms is given in Appendix B. For the last set, we have the following:

$$w_{3,1}^{\mathbf{A}}=\int_{\mathbb{R}} s(t-\tau_{pt})s(t-\tau_{pt})^{*}e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b_{pt}(t-\tau_{pt})\right)}dt=w_1$$

$$w_{3,2}^{\mathbf{A}}=\int_{\mathbb{R}} (t-\tau_{pt})s(t-\tau_{pt})s(t-\tau_{pt})^{*}e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b_{pt}(t-\tau_{pt})\right)}dt=w_2$$

$$w_{3,3}^{\mathbf{A}}=\int_{\mathbb{R}} (t-\tau_{pt})^2 s(t-\tau_{pt})s(t-\tau_{pt})^{*}e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b_{pt}(t-\tau_{pt})\right)}dt=W_{2,2}$$

$$w_{3,4}^{\mathbf{A}}=\int_{\mathbb{R}} s^{(1)}(t-\tau_{pt})s(t-\tau_{pt})^{*}e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b_{pt}(t-\tau_{pt})\right)}dt=w_3$$

$$w_{3,5}^{\mathbf{A}}=\int_{\mathbb{R}} (t-\tau_{pt})s^{(1)}(t-\tau_{pt})s(t-\tau_{pt})^{*}e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b_{pt}(t-\tau_{pt})\right)}dt=w_4$$

$$w_{3,6}^{\mathbf{A}}=\int_{\mathbb{R}} s^{(2)}(t-\tau_{pt})s(t-\tau_{pt})^{*}e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b_{pt}(t-\tau_{pt})\right)}dt=-W_{3,3} \tag{A9}$$

B | DERIVATION OF INTERFERENCE CONVOLUTION TERMS USING FOURIER TRANSFORM PROPERTIES

B.1 | Prior Considerations

First, we evaluate the Fourier transform of a set of functions. Remembering that the signal is band-limited by band $B \leq F_s$, we have the following:

$$s(t) \rightleftharpoons \mathrm{FT}\left\{s(t)\right\}(f) \triangleq S(f)=\left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} s(nT_s)e^{-j2\pi fnT_s}\right)1_{\left[-\frac{F_s}{2};\frac{F_s}{2}\right]} \tag{B1}$$

To address any issue that may arise from the spectral shift due to the Doppler effect, one must simply set F_s to be sufficiently large such that $\frac{F_s}{2} \geq \frac{B}{2}+f_c \max\left\{|b|,|b_{pt}|,|b-b_{pt}|\right\}$.

A first expression is a simple application of the frequency shift relation that is obtained when using the Fourier transform of a signal multiplied by a complex time-varying exponential:

$$s(t)e^{j2\pi f_c bt} \rightleftharpoons \mathrm{FT}\left\{s(t)e^{j2\pi f_c bt}\right\}(f) \triangleq S(f-f_c b) \tag{B2}$$

Then, with s_1 defined as $s_1(t;b)=s(t)e^{j2\pi f_c bt}$, we have the following:

$$ts_1(t;b) \leftrightharpoons \frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\underbrace{\left(\mathrm{FT}\left\{s_1(t;b)\right\}(f)\right)}_{(\text{B2})} \tag{B3}$$

Therefore, we obtain the following:

$$ts(t)e^{j2\pi f_c bt} \leftrightharpoons \frac{j}{2\pi}\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f-f_c b)\right) \tag{B4}$$

Similarly, we obtain the following relation:

$$t^2 s(t)e^{j2\pi f_c bt} \rightleftharpoons \left(\frac{j}{2\pi}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}f^2}\Big(S(f-f_c b)\Big) \tag{B5}$$

With the superscript $^{(1)}$ referring to the first time derivative, we have the following:

$$\begin{aligned} s_1^{(1)}(t;b) &\triangleq \frac{\mathrm{d}}{\mathrm{d}t}(s_1(t;b)) = s^{(1)}(t)e^{j2\pi f_c bt} + (j2\pi f_c b)s_1(t;b) \\ \Leftrightarrow s^{(1)}(t)e^{j2\pi f_c bt} &= s_1^{(1)}(t;b) - (j2\pi f_c b)s_1(t;b) \end{aligned}$$

We have the Fourier transform of the k-th time derivative of a function as follows:

$$\mathrm{FT}\left\{s^{(k)}(t)\right\}(f) \triangleq (j2\pi f)^k S(f) \tag{B6}$$

Thus, one directly obtains the following relation:

$$s^{(1)}(t)e^{j2\pi f_c bt} \rightleftharpoons j2\pi\left(f-f_c b\right)S(f-f_c b) \tag{B7}$$

Now, if s_2 is defined as $s_2(t;b) = ts(t)e^{j2\pi f_c bt}$, we have the following:

$$\begin{aligned} s_2^{(1)}(t;b) &= s_1(t;b) + ts^{(1)}(t)e^{j2\pi f_c bt} + (j2\pi f_c b)s_2(t;b) \\ \Leftrightarrow ts^{(1)}(t)e^{j2\pi f_c bt} &= -\underbrace{s_1(t;b)}_{(\mathrm{B2})} + \underbrace{s_2^{(1)}(t;b)}_{(\mathrm{B6})} - (j2\pi f_c b)\underbrace{s_2(t;b)}_{(\mathrm{B4})} \end{aligned}$$

Therefore, we obtain the following relation:

$$ts^{(1)}(t)e^{j2\pi f_c bt} \rightleftharpoons -S(f-f_c b) - (f-f_c b)\frac{\mathrm{d}}{\mathrm{d}f}\Big(S(f-f_c b)\Big) \tag{B8}$$

Finally, we take s_1 as $s_1(t;b) = s(t)e^{j2\pi f_c bt}$ as follows:

$$\begin{aligned} s_1^{(2)}(t;b) &= s^{(2)}(t)e^{j2\pi f_c bt} + 2(j2\pi f_c b)s^{(1)}(t)e^{j2\pi f_c bt} + (j2\pi f_c b)^2 s_1(t;b) \\ \Leftrightarrow s^{(2)}(t)e^{j2\pi f_c bt} &= \underbrace{s_1^{(2)}(t;b)}_{(\mathrm{B6})} - (j4\pi f_c b)\underbrace{s^{(1)}(t)e^{j2\pi f_c bt}}_{(\mathrm{B7})} + 4\pi^2(f_c b)^2 \underbrace{s_1(t;b)}_{(\mathrm{B2})} \end{aligned}$$

Consequently, we obtain the following:

$$\begin{aligned} s^{(2)}(t)e^{j2\pi f_c bt} &\rightleftharpoons (j2\pi f)^2 S(f-f_c b) + 8\pi^2 f_c b(f-f_c b)S(f-f_c)) + 4\pi^2(f_c b)^2 S(f-f_c) \\ &\rightleftharpoons -4\pi^2\left(f-f_c b\right)^2 S(f-f_c b) \end{aligned} \tag{B9}$$

B.2 | Evaluation of the Integrals

B.2.1 | *Derivation of Integral $w_{1,3}^{\mathbf{A}}(\eta)$*

$$\begin{aligned} w_{1,3}^{\mathbf{A}}(\eta) &= \int_{\mathbb{R}} (t-\tau_{pt})^2 s(t-\tau_{pt})s(t-\tau)^* e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b(t-\tau)\right)}dt \\ &= e^{-j\omega_c b\Delta\tau}\int_{\mathbb{R}} u^2 s(u)s(u-\Delta\tau)^* e^{j\omega_c \Delta bu}\mathrm{d}u \end{aligned}$$

We apply the Fourier transform properties over the Hermitian product:

$$\begin{aligned} w_{1,3}^{\mathbf{A}}(\eta)e^{j\omega_c b\Delta\tau} &= \int_{\mathbb{R}} \underbrace{u^2 s(u)e^{j\omega_c \Delta b u}}_{(B5)} \left(s(u-\Delta\tau)\right)^* \mathrm{d}u \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\left(\frac{j}{2\pi}\right)^2 \frac{\mathrm{d}^2}{\mathrm{d}f^2}\left(S(f-f_c\Delta b)\right)\right)\left(S(f)e^{-j2\pi f\Delta\tau}\right)^* \mathrm{d}f \\ &= \frac{1}{F_s^3}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T\mathbf{D}^2\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)v(f)^*\right) e^{j2\pi f\frac{\Delta\tau}{T_s}}\left(\mathbf{s}^H v(f)\right)\mathrm{d}f \\ &= \frac{1}{F_s^3}\mathbf{s}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)\mathbf{D}^2\mathbf{s} \end{aligned}$$

Hence, we obtain the following:

$$\boxed{w_{1,3}^{\mathbf{A}}(\eta) = \frac{1}{F_s^3}\mathbf{s}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)\mathbf{D}^2\mathbf{s}\, e^{-j\omega_c b\Delta\tau}} \tag{B10}$$

with $\mathbf{U}(p)$ defined in Equation (17) and $\mathbf{V}^{\Delta,0}(q)$ defined in Equation (18). Note the following relations:

$$v(f) = \begin{pmatrix} \dots & e^{j2\pi fn} & \dots \end{pmatrix}^T_{N_1 \le n \le N_2} \tag{B11}$$

$$\mathbf{V}^{\Delta,0}(q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(f)v^H(f)e^{-j2\pi fq}\mathrm{d}f \tag{B12}$$

$$\left[\mathbf{V}^{\Delta,0}(q)\right]_{k,l} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi f(k-l-q)}\mathrm{d}f = \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\sin\left(\pi(k-l-q)\right)}{\pi(k-l-q)} = \mathrm{sinc}\left(k-l-q\right) \tag{B13}$$

B.2.2 | Derivation of Integral $w_{1,5}^{\mathbf{A}}$

$$\begin{aligned} w_{1,5}^{\mathbf{A}}(\eta) &= \int_{\mathbb{R}} (t-\tau_{pt})s^{(1)}(t-\tau_{pt})s(t-\tau)^* e^{-j\omega_c\left(b_{pt}(t-\tau_{pt})-b_k(t-\tau)\right)}\mathrm{d}t \\ &= e^{-j\omega_c b\Delta\tau}\int_{\mathbb{R}} us^{(1)}(u)s(u-\Delta\tau)^* e^{j\omega_c\Delta b u}\mathrm{d}u \end{aligned}$$

Therefore, we have the following relation:

$$\begin{aligned} w_{1,5}^{\mathbf{A}}(\eta)e^{j\omega_c b\Delta\tau} &= \int_{\mathbb{R}} \underbrace{us^{(1)}(u)e^{j\omega_c\Delta b u}}_{(B8)}\left(s(u-\Delta\tau)\right)^* \mathrm{d}u \\ &= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(-S(f-f_c\Delta b) - (f-f_c\Delta b)\frac{\mathrm{d}}{\mathrm{d}f}\left(S(f-f_c\Delta b)\right)\right)\left(S(f)e^{-j2\pi f\Delta\tau}\right)^* \mathrm{d}f \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{U}\left(-\frac{f_c \Delta b}{F_s}\right) v(f)^* \right) e^{j2\pi f \frac{\Delta\tau}{T_s}} \left(\mathbf{s}^H v(f)\right) \mathrm{d}f \\
&\quad + \frac{1}{F_s}\int_{-\frac{1}{2}}^{\frac{1}{2}} j2\pi f \left(\mathbf{s}^T \mathbf{D}\mathbf{U}\left(-\frac{f_c \Delta b}{F_s}\right) v(f)^* \right) e^{j2\pi f \frac{\Delta\tau}{T_s}} \left(\mathbf{s}^H v(f)\right) \mathrm{d}f \\
&\quad - j2\pi \frac{f_c \Delta b}{F_s^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{D}\mathbf{U}\left(-\frac{f_c \Delta b}{F_s}\right) v(f)^* \right) e^{j2\pi f \frac{\Delta\tau}{T_s}} \left(\mathbf{s}^H v(f)\right) \mathrm{d}f \\
&= -\frac{1}{F_s}\mathbf{s}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c \Delta b}{F_s}\right)\mathbf{s} + \frac{1}{F_s}\mathbf{s}^H \mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c \Delta b}{F_s}\right)\mathbf{D}\mathbf{s} \\
&\quad - j2\pi \frac{f_c \Delta b}{F_s^2}\mathbf{s}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c \Delta b}{F_s}\right)\mathbf{D}\mathbf{s}
\end{aligned}
$$

Hence, we obtain the following:

$$
\boxed{
\begin{aligned}
w_{1,5}^{\mathbf{A}}(\eta) = &\left(-\frac{1}{F_s}\mathbf{s}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c \Delta b}{F_s}\right)\mathbf{s} + \frac{1}{F_s}\mathbf{s}^H \mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c \Delta b}{F_s}\right)\mathbf{D}\mathbf{s} \right. \\
&\left. - j\frac{\omega_c \Delta b}{F_s^2}\mathbf{s}^H \mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c \Delta b}{F_s}\right)\mathbf{D}\mathbf{s} \right) e^{-j\omega_c b \Delta\tau}
\end{aligned}
}
\tag{B14}
$$

with $\mathbf{U}$, $\mathbf{V}^{\Delta,0}$, and $\mathbf{V}^{\Delta,1}(q)$ defined in Equations (17), (18), and (14c), respectively. Note that we have the following relations:

$$
\mathbf{V}^{\Delta,1}(q) = j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f v(f) v^H(f) e^{-j2\pi f q} \mathrm{d}f \tag{B15}
$$

$$
\begin{aligned}
\left[\mathbf{V}^{\Delta,1}(q)\right]_{k,l} &= j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f e^{j2\pi f(k-l-q)} \mathrm{d}f = j2\pi \left(\left[\frac{f e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \mathrm{d}f \right) \\
&= \frac{j2\pi}{j2\pi(k-l-q)} \left(\left[\frac{1}{2} e^{j\pi(k-l-q)} - \left(-\frac{1}{2}\right) e^{-j\pi(k-l-q)} \right] - \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \right) \\
&= \frac{1}{k-l-q} \left(\cos\left(\pi(k-l-q)\right) - \operatorname{sinc}\left(k-l-q\right) \right)
\end{aligned}
\tag{B16}
$$

B.2.3 | Derivation of Integral $w_{1,6}^{\mathbf{A}}$

$$
\begin{aligned}
w_{1,6}^{\mathbf{A}}(\eta) &= \int_{\mathbb{R}} s^{(2)}(t-\tau_{pt}) s(t-\tau)^* e^{-j\omega_c \left(b_{pt}(t-\tau_{pt}) - b(t-\tau)\right)} \mathrm{d}t \\
&= e^{-j\omega_c b \Delta\tau} \int_{\mathbb{R}} s^{(2)}(u) s(u-\Delta\tau)^* e^{j\omega_c \Delta b u} \mathrm{d}u
\end{aligned}
$$

Therefore, we have the following:

$$
\begin{aligned}
w_{1,6}^{\mathbf{A}}(\eta)e^{j\omega_c b\Delta\tau} &= \int_{\mathbb{R}} \underbrace{s^{(2)}(u)e^{j\omega_c \Delta b u}}_{(\text{B9})}\left(s(u-\Delta\tau)\right)^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(-4\pi^2(f-f_c\Delta b)^2 S(f-f_c\Delta b)\right)\left(S(f)e^{-j2\pi f\Delta\tau}\right)^* \mathrm{d}f \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(-F_s(4\pi^2 f^2) - j4\pi f_c \Delta b(j2\pi f) - 4\pi^2\frac{(f_c\Delta b)^2}{F_s}\right)\left(\mathbf{s}^T\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)\nu(f)^*\right)\right) \\
&\quad \times e^{j2\pi f\frac{\Delta\tau}{T_s}}\left(\mathbf{s}^H\nu(f)\right)\mathrm{d}f \\
&= -F_s\mathbf{s}^H\mathbf{V}^{\Delta,2}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)\mathbf{s} - j4\pi f_c\Delta b\mathbf{s}^H\mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)\mathbf{s} \\
&\quad - 4\pi^2\frac{(f_c\Delta b)^2}{F_s}\mathbf{s}^H\mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)\mathbf{s}
\end{aligned}
$$

Hence, we have the following relation:

$$
\boxed{
\begin{aligned}
w_6^{\mathbf{A}}(\eta) = \Bigg(&-F_s\mathbf{s}^H\mathbf{V}^{\Delta,2}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)\mathbf{s} - j2\omega_c\Delta b\mathbf{s}^H\mathbf{V}^{\Delta,1}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)\mathbf{s} \\
&-\frac{(\omega_c\Delta b)^2}{F_s}\mathbf{s}^H\mathbf{V}^{\Delta,0}\left(-\frac{\Delta\tau}{T_s}\right)\mathbf{U}\left(-\frac{f_c\Delta b}{F_s}\right)\mathbf{s}\Bigg)e^{-j\omega_c b\Delta\tau}
\end{aligned}
}
\tag{B17}
$$

with $\mathbf{U}(\cdot)$ defined in Equation (17), $\mathbf{V}^{\Delta,0}(\cdot)$ defined in Equation (18), $\mathbf{V}^{\Delta,1}$ defined in Equation (14c), and $\mathbf{V}^{\Delta,2}(\cdot)$ defined in Equation (14d). Note the following relations:

$$
\mathbf{V}^{\Delta,2}(q) = 4\pi^2\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2\nu(f)\nu^H(f)e^{-j2\pi fq}\mathrm{d}f \tag{B18}
$$

$$
\begin{aligned}
\left[\mathbf{V}^{\Delta,2}(q)\right]_{k,l} &= 4\pi^2\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 e^{j2\pi f(k-l-q)}\mathrm{d}f = 4\pi^2\left(\left[\frac{f^2e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}}\frac{2fe^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\mathrm{d}f\right) \\
&= \frac{4\pi^2}{j2\pi(k-l-q)}\frac{1}{4}\left[e^{j\pi(k-l-q)} - e^{-j\pi(k-l-q)}\right] \\
&\quad - \frac{8\pi^2}{j2\pi(k-l-q)}\left(\left[\frac{fe^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{+\frac{1}{2}}\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\mathrm{d}f\right) \\
&= \pi^2\mathrm{sinc}\left(k-l-q\right) - \frac{8\pi^2}{\left(j2\pi(k-l-q)\right)^2} \\
&\quad \times\left(\left[\frac{1}{2}e^{j\pi(k-l-q)} - \left(-\frac{1}{2}\right)e^{-j\pi(k-l-q)}\right] - \left[\frac{e^{j2\pi f(k-l-q)}}{j2\pi(k-l-q)}\right]_{-\frac{1}{2}}^{\frac{1}{2}}\right) \\
&= \pi^2\mathrm{sinc}\left(k-l-q\right) + \frac{8\pi^2}{4\pi^2(k-l-q)^2}\times\left(\cos\left(\pi(k-l-q)\right) - \mathrm{sinc}\left(k-l-q\right)\right) \\
&= \pi^2\mathrm{sinc}\left(k-l-q\right) + 2\frac{\cos\left(\pi(k-l-q)\right) - \mathrm{sinc}\left(k-l-q\right)}{\left(k-l-q\right)^2}
\end{aligned}
\tag{B19}
$$

B.2.4 | Derivation of Integral $w_{2,1}^{\mathbf{A}}$

$$w_{2,1}^{\mathbf{A}} = \int_{\mathbb{R}} s(t-\tau_{pt}) I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})} dt = \int_{\mathbb{R}} s(u) I(u+\tau_{pt})^* e^{-j\omega_c b_{pt} u} \mathrm{d}u$$

We apply the Fourier transform properties over the Hermitian product:

$$\begin{aligned}
w_{2,1}^{\mathbf{A}} &= \int_{\mathbb{R}} \underbrace{s(u) e^{-j\omega_c b_{pt} u}}_{(B2)} \left(I(u+\tau_{pt}) \right)^* \mathrm{d}u = \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} S(f + f_c b_{pt}) \left(I(f) e^{j2\pi f \tau_{pt}} \right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi(f+f_c b_{pt})nT_s} \right) e^{-j2\pi f \tau_{pt}} \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n T_s} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{-j2\pi \frac{f_c b_{pt}}{F_s} n} \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{v}(f)^* \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\mathbf{I}^H \mathbf{v}(f) \right) \mathrm{d}f \\
&= \frac{1}{F_s} \mathbf{I}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{v}(f) \mathbf{v}^H(f) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s} = \frac{1}{F_s} \mathbf{I}^H \mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s} \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s}
\end{aligned}$$

Hence, we obtain the following:

$$\boxed{w_{2,1}^{\mathbf{A}} = \frac{1}{F_s} \mathbf{I}^H \mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s} \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s}} \tag{B20}$$

B.2.5 | Derivation of Integral $w_{2,2}^{\mathbf{A}}$

$$w_{2,2}^{\mathbf{A}} = \int_{\mathbb{R}} (t-\tau_{pt}) s(t-\tau_{pt}) I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})} dt = \int_{\mathbb{R}} u s(u) I(u+\tau_{pt})^* e^{-j\omega_c b_{pt} u} \mathrm{d}u$$

Therefore, we have the following:

$$\begin{aligned}
w_{2,2}^{\mathbf{A}} &= \int_{\mathbb{R}} \underbrace{u s(u) e^{-j\omega_c b_{pt} u}}_{(B4)} \left(I(u+\tau_{pt}) \right)^* \mathrm{d}u = \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{j}{2\pi} \frac{\mathrm{d}}{\mathrm{d}f} \left(S(f + f_c b_{pt}) \right) \right) \left(I(f) e^{j2\pi f \tau_{pt}} \right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\frac{1}{F_s} \frac{j}{2\pi} (-j2\pi T_s) \sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi(f+f_c b_{pt})nT_s} \right) e^{-j2\pi f \tau_{pt}} \\
&\quad \times \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n T_s} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi f n} e^{-j2\pi \frac{f_c b_{pt}}{F_s} n} \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{D} \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{v}(f)^* \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\mathbf{I}^H \mathbf{v}(f) \right) \mathrm{d}f
\end{aligned}$$

$$= \frac{1}{F_s^2}\mathbf{I}^H\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} v(f)v^H(f)e^{-j2\pi f\frac{\tau_{pt}}{T_s}}\mathrm{d}f\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{Ds}$$

$$= \frac{1}{F_s^2}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{Ds}$$

Hence, we have the following relation:

$$\boxed{w_{2,2}^{\mathbf{A}} = \frac{1}{F_s^2}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{Ds}} \tag{B21}$$

with $\mathbf{U}$ and $\mathbf{V}^{\Delta,0}$ defined in Equations (17) and (18), respectively, and $\mathbf{D}$ defined in Equation (14b).

B.2.6 | Derivation of Integral $w_{2,3}^{\mathbf{A}}$

$$w_{2,3}^{\mathbf{A}} = \int_{\mathbb{R}} (t-\tau_{pt})^2 s(t-\tau_{pt}) I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt = \int_{\mathbb{R}} u^2 s(u) I(u+\tau_{pt})^* e^{-j\omega_c b_{pt} u}\mathrm{d}u$$

Therefore, we obtain the following:

$$w_{2,3}^{\mathbf{A}} = \int_{\mathbb{R}} \underbrace{u^2 s(u)e^{-j\omega_c b_{pt}u}}_{(\text{B5})}\left(I(u+\tau_{pt})\right)^*\mathrm{d}u$$

$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}}\left(\left(\frac{j}{2\pi}\right)^2\frac{\mathrm{d}^2}{\mathrm{d}f^2}\left(S(f+f_c b_{pt})\right)\right)\left(I(f)e^{j2\pi f\tau_{pt}}\right)^*\mathrm{d}f$$

$$= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}}\left(\frac{1}{F_s}\left(\frac{j}{2\pi}\right)^2(-j2\pi T_s)^2\sum_{n=N_1}^{N_2} s(nT_s)n^2 e^{-j2\pi(f+f_c b_{pt})nT_s}\right)e^{-j2\pi f\tau_{pt}}$$

$$\times\left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} I(nT_s)e^{-j2\pi f nT_s}\right)^*\mathrm{d}f$$

$$= \frac{1}{F_s^3}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sum_{n=N_1}^{N_2} s(nT_s)n^2 e^{-j2\pi fn}e^{-j2\pi\frac{f_c b_{pt}}{F_s}n}\right)e^{-j2\pi f\frac{\tau_{pt}}{T_s}}\left(\sum_{n=N_1}^{N_2} I(nT_s)e^{-j2\pi fn}\right)^*\mathrm{d}f$$

$$= \frac{1}{F_s^3}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\mathbf{s}^T\mathbf{D}^2\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)v(f)^*\right)e^{-j2\pi f\frac{\tau_{pt}}{T_s}}\left(\mathbf{I}^H v(f)\right)\mathrm{d}f$$

$$= \frac{1}{F_s^3}\mathbf{I}^H\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} v(f)v^H(f)e^{-j2\pi f\frac{\tau_{pt}}{T_s}}\mathrm{d}f\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{D}^2\mathbf{s} = \frac{1}{F_s^3}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{D}^2\mathbf{s}$$

Hence, we have the following relation:

$$\boxed{w_{2,3}^{\mathbf{A}} = \frac{1}{F_s^3}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{D}^2\mathbf{s}} \tag{B22}$$

with $\mathbf{U}$, $\mathbf{V}^{\Delta,0}$, and $\mathbf{D}$ defined in Equations (17), (18), and (14b), respectively.

B.2.7 | Derivation of Integral $w^{\mathbf{A}}_{2,4}$

$$w^{\mathbf{A}}_{2,4} = \int_{\mathbb{R}} s^{(1)}(t-\tau_{pt}) I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})} dt = \int_{\mathbb{R}} s^{(1)}(u) I(u+\tau_{pt})^* e^{-j\omega_c b_{pt} u} \mathrm{d}u$$

Therefore, we have the following:

$$\begin{aligned}
w^{\mathbf{A}}_{2,4} &= \int_{\mathbb{R}} \underbrace{s^{(1)}(u) e^{-j\omega_c b_{pt} u}}_{(B7)} \left(I(u+\tau_{pt}) \right)^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(j2\pi (f + f_c b_{pt}) S(f + f_c b_{pt}) \right) \left(I(f) e^{j2\pi f \tau_{pt}} \right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(j2\pi (f + f_c b_{pt}) \frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi (f + f_c b_{pt}) n T_s} \right) e^{-j2\pi f \tau_{pt}} \\
&\quad \times \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n T_s} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(j2\pi (f F_s + f_c b_{pt}) \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{-j2\pi \frac{f_c b_{pt}}{F_s} n} \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \\
&\quad \times \left(\sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f \\
&= \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(j2\pi (f F_s + f_c b_{pt}) \mathbf{s}^T \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) v(f)^* \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\mathbf{I}^H v(f) \right) \mathrm{d}f \\
&= \mathbf{I}^H \left(j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f v(f) v^H(f) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s} \\
&\quad + \frac{j2\pi f_c b_{pt}}{F_s} \mathbf{I}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} v(f) v^H(f) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s} \\
&= \mathbf{I}^H \mathbf{V}^{\Delta,1}\left(\frac{\tau_{pt}}{T_s} \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s} + \frac{j2\pi f_c b_{pt}}{F_s} \mathbf{I}^H \mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s} \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s}
\end{aligned}$$

Hence, we obtain the following relation:

$$\boxed{w^{\mathbf{A}}_{2,4} = \mathbf{I}^H \mathbf{V}^{\Delta,1}\left(\frac{\tau_{pt}}{T_s} \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s} + \frac{j\omega_c b_{pt}}{F_s} \mathbf{I}^H \mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s} \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s}} \tag{B23}$$

with $\mathbf{U}$, $\mathbf{V}^{\Delta,0}$, and $\mathbf{V}^{\Delta,1}$ defined in Equations (17), (18), and (14c), respectively.

B.2.8 | Derivation of Integral $w^{\mathbf{A}}_{2,5}$

$$w^{\mathbf{A}}_{2,5} = \int_{\mathbb{R}} (t-\tau_{pt}) s^{(1)}(t-\tau_{pt}) I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})} dt = \int_{\mathbb{R}} u s^{(1)}(u) I(u+\tau_{pt})^* e^{-j\omega_c b_{pt} u} \mathrm{d}u$$

Therefore, we have the following:

$$
\begin{aligned}
w_{2,5}^{\mathbf{A}} &= \int_{\mathbb{R}} \underbrace{u s^{(1)}(u) e^{-j\omega_c b_{pt} u}}_{\text{(B8)}} \left(I(u+\tau_{pt}) \right)^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(-S(f+f_c b_{pt}) - (f+f_c b_{pt}) \frac{\mathrm{d}}{\mathrm{d}f} \left(S(f+f_c b_{pt}) \right) \right) \left(I(f) e^{j2\pi f \tau_{pt}} \right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \Bigg(-\left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi (f+f_c b_{pt}) nT_s} \right) \\
&\quad - (f+f_c b_{pt}) \left(\frac{1}{F_s} (-j2\pi T_s) \sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi (f+f_c b_{pt}) nT_s} \right) \Bigg) \\
&\quad \times e^{-j2\pi f \tau_{pt}} \left(\frac{1}{F_s} \sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n T_s} \right)^* \mathrm{d}f \\
&= -\frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=N_1}^{N_2} s(nT_s) e^{-j2\pi f n} e^{-j2\pi \frac{f_c b_{pt}}{F_s} n} \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f \\
&\quad + \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} j2\pi f \left(\sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi f n} e^{-j2\pi \frac{f_c b_{pt}}{F_s} n} \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f \\
&\quad + j2\pi \frac{f_c b_{pt}}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=N_1}^{N_2} s(nT_s) n e^{-j2\pi f n} e^{-j2\pi \frac{f_c b_{pt}}{F_s} n} \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\sum_{n=N_1}^{N_2} I(nT_s) e^{-j2\pi f n} \right)^* \mathrm{d}f \\
&= -\frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \nu(f)^* \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\mathbf{I}^H \nu(f) \right) \mathrm{d}f \\
&\quad + \frac{1}{F_s} \int_{-\frac{1}{2}}^{\frac{1}{2}} j2\pi f \left(\mathbf{s}^T \mathbf{D} \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \nu(f)^* \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\mathbf{I}^H \nu(f) \right) \mathrm{d}f \\
&\quad + j2\pi \frac{f_c b_{pt}}{F_s^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\mathbf{s}^T \mathbf{D} \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \nu(f)^* \right) e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \left(\mathbf{I}^H \nu(f) \right) \mathrm{d}f \\
&= -\frac{1}{F_s} \mathbf{I}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \nu(f) \nu(f)^H e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s} \\
&\quad + \frac{1}{F_s} \mathbf{I}^H \left(j2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} f \nu(f) \nu(f)^H e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{D}\mathbf{s} \\
&\quad + j2\pi \frac{f_c b_{pt}}{F_s^2} \mathbf{I}^H \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \nu(f) \nu(f)^H e^{-j2\pi f \frac{\tau_{pt}}{T_s}} \mathrm{d}f \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{D}\mathbf{s} \\
&= -\frac{1}{F_s} \mathbf{I}^H \mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s} \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{s} + \frac{1}{F_s} \mathbf{I}^H \mathbf{V}^{\Delta,1}\left(\frac{\tau_{pt}}{T_s} \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{D}\mathbf{s} \\
&\quad + j2\pi \frac{f_c b_{pt}}{F_s^2} \mathbf{I}^H \mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s} \right) \mathbf{U}\left(\frac{f_c b_{pt}}{F_s} \right) \mathbf{D}\mathbf{s}
\end{aligned}
$$

Hence, we obtain the following relation:

$$\boxed{\begin{aligned} w_{2,5}^{\mathbf{A}} = &-\frac{1}{F_s}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s}+\frac{1}{F_s}\mathbf{I}^H\mathbf{V}^{\Delta,1}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{D}\mathbf{s} \\ &+j\frac{\omega_c b_{pt}}{F_s^2}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{D}\mathbf{s}\end{aligned}} \tag{B24}$$

with $\mathbf{U}$, $\mathbf{V}^{\Delta,0}$, $\mathbf{V}^{\Delta,1}$, and $\mathbf{D}$ defined in Equations (17), (18), (14c), and (14b), respectively.

B.2.9 | Derivation of Integral $w_{2,6}^{\mathbf{A}}$

$$w_{2,6}^{\mathbf{A}} = \int_{\mathbb{R}} s^{(2)}(t-\tau_{pt})I(t)^* e^{-j\omega_c b_{pt}(t-\tau_{pt})}dt = \int_{\mathbb{R}} s^{(2)}(u)I(u+\tau_{pt})^* e^{-j\omega_c b_{pt}u}\mathrm{d}u$$

Therefore, we obtain the following:

$$\begin{aligned}
w_{2,6}^{\mathbf{A}} &= \int_{\mathbb{R}} \underbrace{s^{(2)}(u)e^{-j\omega_c b_{pt}u}}_{(B9)}\left(I(u+\tau_{pt})\right)^* \mathrm{d}u \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(-4\pi^2(f+f_c b_{pt})^2 S(f+f_c b_{pt})\right)\left(I(f)e^{j2\pi f\tau_{pt}}\right)^* \mathrm{d}f \\
&= \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} \left(\left(-4\pi^2 f^2 - 8\pi^2 ff_c b_{pt} - 4\pi^2(f_c b_{pt})^2\right)\left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} s(nT_s)e^{-j2\pi(f+f_c b_{pt})nT_s}\right)\right) \\
&\quad \times e^{j2\pi f\tau_{pt}}\left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} I(nT_s)e^{-j2\pi fnT_s}\right)^* \mathrm{d}f \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(-4\pi^2(fF_s)^2 - 8\pi^2(fF_s)f_c b_{pt} - 4\pi^2(f_c b_{pt})^2\right)\left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} s(nT_s)e^{-j2\pi fn}e^{-j2\pi\frac{f_c b_{pt}}{F_s}n}\right)\right) \\
&\quad \times e^{-j2\pi f\frac{\tau_{pt}}{T_s}}\left(\frac{1}{F_s}\sum_{n=N_1}^{N_2} I(nT_s)e^{-j2\pi fn}\right)^* \mathrm{d}fF_s \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(-F_s(4\pi^2 f^2) + j4\pi f_c b_{pt}(j2\pi f) - 4\pi^2\frac{(f_c b_{pt})^2}{F_s}\right)\left(\mathbf{s}^T\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)v(f)^*\right)\right) \\
&\quad \times e^{-j2\pi f\frac{\tau_{pt}}{T_s}}\left(\mathbf{I}^H v(f)\right)\mathrm{d}f \\
&= -F_s\mathbf{I}^H\left(4\pi^2\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2 v(f)v(f)^H e^{-j2\pi f\frac{\tau_{pt}}{T_s}}\mathrm{d}f\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s} \\
&\quad + j4\pi f_c b_{pt}\mathbf{I}^H\left(j2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} fv(f)v(f)^H e^{-j2\pi f\frac{\tau_{pt}}{T_s}}\mathrm{d}f\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s} \\
&\quad - 4\pi^2\frac{(f_c b_{pt})^2}{F_s}\mathbf{I}^H\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} v(f)v(f)^H e^{-j2\pi f\frac{\tau_{pt}}{T_s}}\mathrm{d}f\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s} \\
&= -F_s\mathbf{I}^H\mathbf{V}^{\Delta,2}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s} + j4\pi f_c b_{pt}\mathbf{I}^H\mathbf{V}^{\Delta,1}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s} \\
&\quad - 4\pi^2\frac{(f_c b_{pt})^2}{F_s}\mathbf{I}^H\mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s}
\end{aligned}$$

Hence, we have the following relation:

$$w_{2,6}^{\mathbf{A}} = -F_s \mathbf{I}^H \mathbf{V}^{\Delta,2}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s} + j2\omega_c b_{pt} \mathbf{I}^H \mathbf{V}^{\Delta,1}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s} - \frac{(\omega_c b_{pt})^2}{F_s}\mathbf{I}^H \mathbf{V}^{\Delta,0}\left(\frac{\tau_{pt}}{T_s}\right)\mathbf{U}\left(\frac{f_c b_{pt}}{F_s}\right)\mathbf{s} \tag{B25}$$

with $\mathbf{U}$, $\mathbf{V}^{\Delta,0}$, and $\mathbf{V}^{\Delta,1}$ defined in Equations (17), (18), and (14c), respectively.

B.3 | Matrix Properties

Based on the definitions of matrices $\mathbf{V}^{\Delta,0}$, $\mathbf{V}^{\Delta,1}$, $\mathbf{V}^{\Delta,2}$, and $\mathbf{U}$, we have the following relations:

- $\left(\mathbf{V}^{\Delta,0}(q)\right)^H = \mathbf{V}^{\Delta,0}(-q)$
- $\left(\mathbf{V}^{\Delta,1}(q)\right)^H = -\mathbf{V}^{\Delta,1}(-q)$
- $\left(\mathbf{V}^{\Delta,2}(q)\right)^H = \mathbf{V}^{\Delta,2}(-q)$
- $\left(\mathbf{U}(p)\right)^H = \mathbf{U}(-p)$