4th SLSIP Workshop

Estimation of Extended Target Impulse Response of Unknown Size

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Outline

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Intuition 1 – Urban Scenario

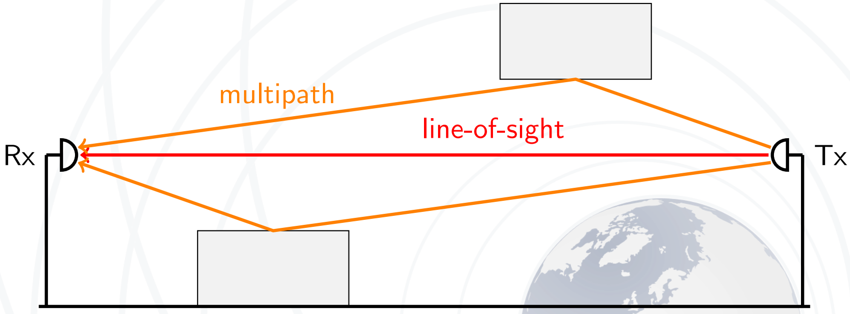

Figure: Typical Urban scenario: propagation channel estimation for equalization

Intuition 1 – Urban Scenario

The received signal is the sum of the line-of-sight signal (LOS) and several echoes:

$$x(t) = \sum_{p=1}^{3} \alpha_p s(t - \tau_p) + n(t), \quad (1)$$

which can also be put in a matrix form:

$$\mathbf{x} = \mathbf{A}\alpha + \mathbf{n}, \quad (2)$$

with $\mathbf{A} = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3]$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T$.
The complexity of the likelihood to be maximized increases with the number of multipaths.

Intuition 2 – Extended Target Scenario

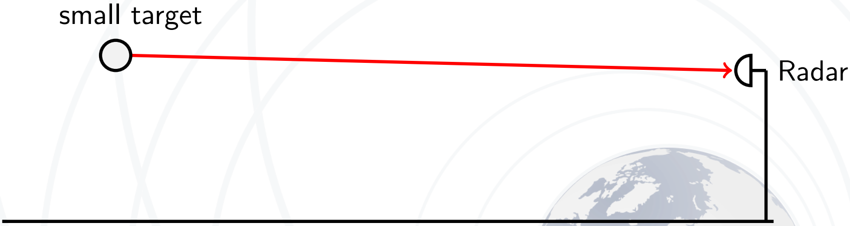

Figure: Extended Target scenario

Intuition 2 – Extended Target Scenario

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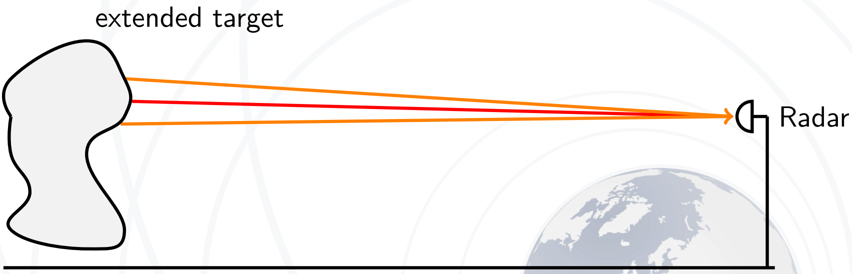

Figure: Extended Target scenario

Intuition 2 – Extended Target Scenario

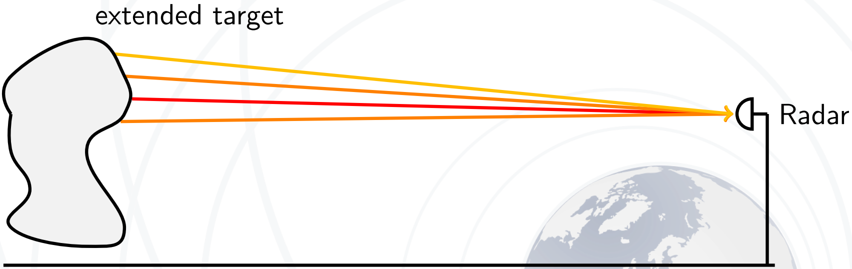

Figure: Extended Target scenario

Intuition 2 – Extended Target Scenario

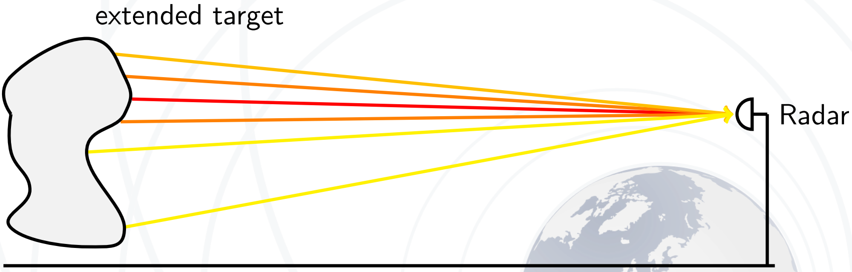

Figure: Extended Target scenario

Intuition 2 – Extended Target Scenario

Again the received signal is the sum of P several echoes that recombined in the antenna.

$$x(t) = \sum_{p=1}^{P} \alpha_p s(t - \tau_p) + n(t), \tag{3}$$

A way to deal with this problem was previously studied*. The idea is to assume that the echoes are regularly spaced in time:

$$\tau_p = \tau_0 + pT_s \tag{4}$$

This regular period could be fixed by the sampling frequency. Therefore, the complexity of the likelihood to be maximized stays the same: there is only one time-delay to estimate: τ_0.

*[1] Zhao and Huang, "Cramer-Rao Lower Bounds for the Joint Delay Doppler Estimation of an Extended Target," 2016.

Signal Model

Signal model with Doppler frequency F_d at the input of the RF front-end:

$$x(t) = \sum_{p=1}^{P} \alpha_p s(t - \tau_p) e^{j2\pi(f_c + F_d)(t-\tau)} + n(t), \qquad (5)$$

with $\tau_p = \tau_0 + pT_s$.
The received sampled signal can be written under a Conditional Signal Model form:

$$\mathbf{x} = \mathbf{A}_P(\boldsymbol{\tau}_0, F_d)\alpha + \mathbf{n} \qquad (6)$$

Signal Model

$$\mathbf{x} = \mathbf{A}_P(\boldsymbol{\tau}_0, F_d)\boldsymbol{\alpha} + \mathbf{n},\ \ \mathbf{n} \sim \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_N), \tag{7}$$

with

$$\begin{aligned}
\mathbf{x}^T &= (\ldots, x(nT_s), \ldots), && (8)\\
\mathbf{A}_P(\tau_0, F_d) &= \left[\ldots, s_p(\tau_0, F_d), \ldots\right], && (9)\\
\mathbf{s}_p(\tau_0, F_d)^T &= \left(\ldots, s(nT_s - \tau_p)e^{-j2\pi F_d(nT_s - \tau_p)}, \ldots\right), && (10)\\
\boldsymbol{\alpha}^T &= \left(\ldots, \rho_p e^{j\phi_p}, \ldots\right), && (11)\\
\mathbf{n}^T &= (\ldots, n(nT_s), \ldots). && (12)
\end{aligned}$$

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Cramér-Rao Bounds (CRB)

Using the Slepian-Bangs formula*, derivation of the CRB for the estimation of $\boldsymbol{\epsilon}^T = [\sigma_n^2, \tau_0, F_d, \boldsymbol{\rho}, \boldsymbol{\phi}]$:

$$\mathbf{F}_{\boldsymbol{\epsilon}|\boldsymbol{\epsilon}} = \begin{bmatrix} F_{\sigma_n^2|\boldsymbol{\epsilon}} & \mathbf{0}_{1,2P+2} \\ \mathbf{0}_{2P+2,1} & \mathbf{F}_{\bar{\boldsymbol{\epsilon}}|\boldsymbol{\epsilon}} \end{bmatrix} \quad (13)$$

$$\mathbf{F}_{\bar{\boldsymbol{\epsilon}}|\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) = \frac{2F_s}{\sigma_n^2} \mathrm{Re}\left\{\mathbf{Q}\mathbf{W}^{\delta}\mathbf{Q}^H\right\} \quad (14)$$

where $\mathbf{W}^{\delta}$ is a matrix that depends on the signal baseband samples.

*[2] Yau and Bresler, "A Compact Cramér-Rao Bound Expression for Parametric Estimation of Superimposed Signals," 1992.

Maximum Likelihood Estimator (MLE)

- To validate the CRB expression, need of an efficient estimator,
- for the considered signal model, the MLE is asymptotically efficient (in our case: asymptotic = SNR large),
- thanks to the time constraint between each pulse, the MLE reduces to a 2-dimensional search:

$$\left(\widehat{\tau_0}, \widehat{F_d}\right) = \arg\max_{\tau_0, F_d} \left\| \mathbf{P}_{\mathbf{A}_P}(\tau_0, F_d)\mathbf{x} \right\|^2 \quad (15)$$

$$\widehat{\rho_p} = \left| \left[\left(\mathbf{A}_P^H \mathbf{A}_P \right)^{-1} \mathbf{A}_P^H \mathbf{x} \right]_p \right| \quad (16)$$

$$\widehat{\phi_p} = \arg \left\{ \left[\left(\mathbf{A}_P^H \mathbf{A}_P \right)^{-1} \mathbf{A}_P^H \mathbf{x} \right]_p \right\} \quad (17)$$

Outline

Scenario Definition

Simulation set-up

- GPS L1 C/A (BPSK(1))
- $F_s = 4$ MHz
- Number of pulses (known): $P = 4$
- inter-pulse time interval: $\Delta\tau_p = 1/F_s$
- 1000 Monte Carlo runs to estimate the MSE of IR-MLE$(P, \Delta\tau_p)$.

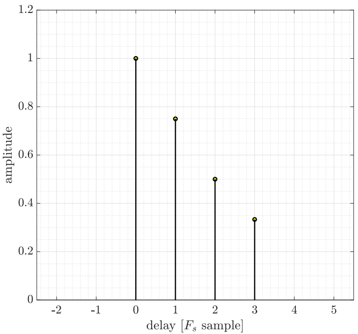


Figure: Scenario definition.

Numerical Results (1)

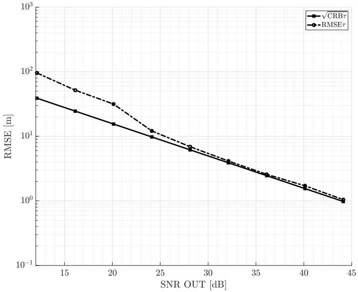


Figure: τ

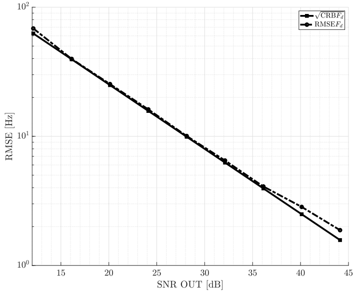


Figure: F_d

Numerical Results (2)

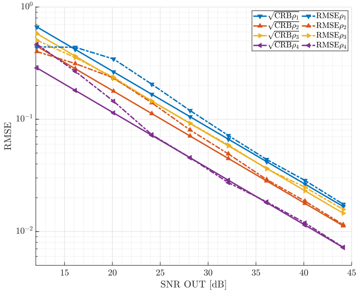


Figure: ρ

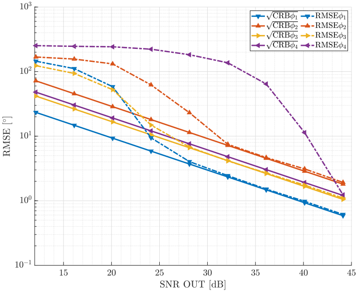


Figure: ϕ

Outline

Detection Problem: Iterative Technique

A first technique to estimate the number of pulses is an iterative technique:

- assume that P sources have been estimated.
- the $P+1$ pulse is tested to decide whether it is part of the impulse response of part of noise.
- the test statistics is built based on a likelihood criterion:

$$T_{P+\text{next}} = \left| \left(\mathbf{P}^{\perp}_{\mathbf{A}_P(\widehat{\tau_0}, \widehat{F_d})} \mathbf{x} \right)^H \mathbf{s}_{P+1}(\widehat{\tau_0}, \widehat{F_d}) \right|^2 \gtrless \text{threshold } t \qquad (18)$$

- it can be shown that, under the null hypothesis (no energy in the next pulse), when normalized, $T_{P+\text{next}} \sim \chi^2_2$, which allows to fix the threshold t for a given probability of false alarm (PFA).

Detection Problem: Overshoot-and-Decimate Technique

- Iterative techniques: high computational load,
- Other approach: overshoot the number of sources and filter out the least relevant estimates*,
  - true value is K, estimation of $M > K$ sources.
  - for each of the M estimates, evaluation of a likelihood ratio:

$$\mathrm{LR}_m = \frac{\left\| \mathbf{P}^{\perp}_{\mathbf{A}_M} \mathbf{x} \right\|^2}{\left\| \mathbf{P}^{\perp}_{\mathbf{A}_{M-1,m}} \mathbf{x} \right\|^2} \gtrless \text{threshold } t \tag{19}$$

  - it can be shown that, under the null hypothesis (the m-th estimates is not relevant), $\mathrm{LR}_m \sim \beta_{1,N-1}$, which allows to fix the threshold t for a given PFA.

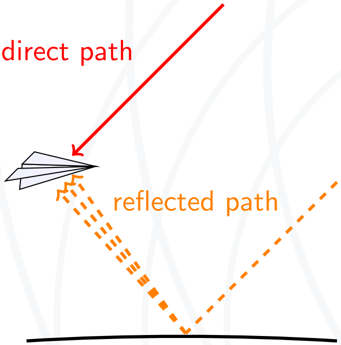
*[3] Chung et al "Detection of the Number of Signals Using the Benjamini-Hochberg Procedure," 2007.

Outline

Conclusion

- Wrap-up of this presentation:
  - Extended target signal model
  - Closed-form CRB derivation
  - MLE implementation and CRB validation
  - Size determination based on hypothesis tests.
- Perspectives:
  - Signal model: the impulse response as a random object?
  - Size determination techniques: information theoretic criterion (AIC, MDL) or subspace consideration (SSM),
  - classification of impulse responses,
  - application to real data.

A Possible Application: GNSS-R



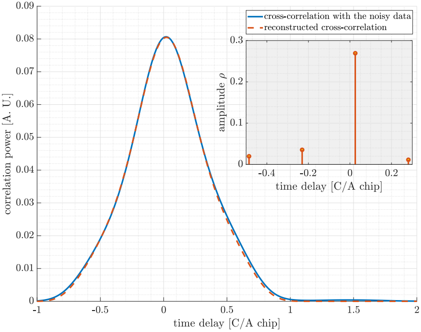


Figure: Direct signal reconstructed CCF based on the estimated IR. Baltic Sea experiment, ICE-IEEC.

Et voilà

Thank you for your attention!

References I

[1] T. Zhao and T. Huang, "Cramer-Rao Lower Bounds for the Joint Delay-Doppler Estimation of an Extended Target," *IEEE Transactions on Signal Processing*, vol. 64, no. 6, pp. 1562–1573, 2016.

[2] S. F. Yau and Y. Bresler, "A Compact Cramér-Rao Bound Expression for Parametric Estimation of Superimposed Signals," *IEEE Transactions on Signal Processing*, vol. 40, no. 5, pp. 1226–1230, May 1992.

[3] P.-J. Chung, J. F. Bohme, C. F. Mecklenbräuker, and A. O. Hero, "Detection of the Number of Signals Using the Benjamini-Hochberg Procedure," *IEEE Transactions on Signal Processing*, vol. 55, no. 6, pp. 2497–2508, 2007.

Knowing the probability density function of the signal, the likelihood function is:

$$p(\mathbf{x};\boldsymbol{\epsilon}) = \frac{1}{\left(\pi\sigma_n^2\right)^N} e^{-\frac{1}{\sigma_n^2}\|\mathbf{x}-\mathbf{A}_P\alpha\|^2}, \tag{20}$$

maximizing this likelihood is equivalent to minimizing the norm in the exponential function. Then using the orthogonal projector: $\mathbf{P}_{\mathbf{A}_P} = \mathbf{A}_P\left(\mathbf{A}_P^H\mathbf{A}_P\right)^{-1}\mathbf{A}_P^H$, $\mathbf{P}_{\mathbf{A}_P}^{\perp} = \mathbf{I} - \mathbf{P}_{\mathbf{A}_P}$:

$$\begin{aligned}\|\mathbf{x}-\mathbf{A}_P\alpha\|^2 &= \left\|\mathbf{P}_{\mathbf{A}_P}\left(\mathbf{x}-\mathbf{A}_P\alpha\right)\right\|^2 + \left\|\mathbf{P}_{\mathbf{A}_P}^{\perp}\left(\mathbf{x}-\mathbf{A}_P\alpha\right)\right\|^2 \\ &= \left\|\mathbf{A}_P\left(\left(\mathbf{A}_P^H\mathbf{A}_P\right)^{-1}\mathbf{A}_P^H\mathbf{x}-\alpha\right)\right\|^2 + \left\|\mathbf{P}_{\mathbf{A}_P}^{\perp}\mathbf{x}\right\|^2.\end{aligned} \tag{21}$$

back-up: Overshooting Rational

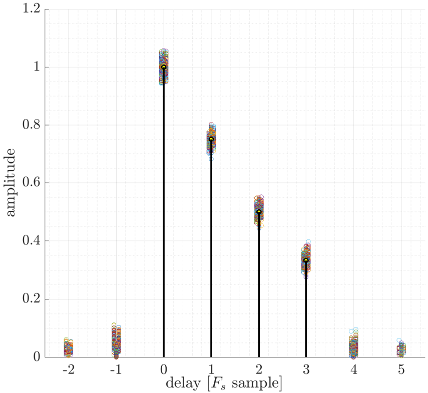


Figure: Example of overshooting: true value is $K = 4$ and number of pulses searched is $M = 6$.